Ultimate strength of box girders by finite element method, PhD dissertation, 1975

Yilmaz, Cetin
1975
ULTIMATE STRENGTH OF BOX GIRDERS BY FINITE ELEMENT METHOD

by

Cetin Yilmaz

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A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Civil Engineering

1975
Approved and recommended for acceptance as a dissertation
in partial fulfillment of the requirements for the degree of Doctor
of Philosophy.

March 18, 1975
(date)

Accepted March 18, 1975

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ACKNOWLEDGMENTS

The author is deeply indebted to Professor Ben T. Yen, Professor in charge of this dissertation for his guidance and continuous encouragement in the preparation of this thesis. The guidance of Professors Roger G. Slutter, Chairman, Celal N. Kostem, George C. Sih and David A. VanHorn, members of the special committee directing the author's doctoral work, is gratefully acknowledged.

Sincere appreciation is expressed to all the author's associates in Fritz Engineering Laboratory. Two of the author's friends need to be mentioned specifically. Dr. Suresh Desai provided the starting point for the computer program developed in this study and Dr. Dirk P. duPlessis for the frequent discussions.

Special thanks are extended to Mrs. Dorothy Fielding for typing the manuscript with great care.
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This dissertation presents the results of an analysis of the behavior of straight rectangular box girders to failure when the loads are applied either through diaphragms or from the web flange junction.

In the analysis the finite element method is used and material nonlinearities are considered.

The basic formulation was developed for the elastic analysis using five degrees of freedom per nodal point and rectangular elements. The load deflection behavior of some tested box girders was closely predicted when the geometric nonlinearities were small. Ultimate load carrying capacity was obtained and was compared with test results. The method also provided an accurate means of analysis for the stresses and deflections in both the elastic range and the inelastic range of material behavior.

A two degree of freedom formulation was adopted in the nonlinear analysis, so that the application of two dimensional failure criterion was possible. The incremental tangent stiffness method and the incremental theory of plasticity were selected for the solution of the nonlinear problem. The load deflection behavior of some tested box girders was closely predicted when the geometric nonlinearities were small.

The incremental tangent stiffness method and shear lag effect were automatically considered. The results were compared with the available test data. The method was used for the analysis of four different examples. A number of examples of thin walled structures showed that a two degree of freedom formulation gave results comparable to those of a five degree of freedom formulation but with relatively less cost. The boundary conditions of box girders were closely modeled and shear deformation and shear lag effect were automatically considered. The results were compared with the available test data.

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1. INTRODUCTION

1.1 Background

The use of thin walled steel box girders and steel composite box girders as main load carrying members in bridge structures has gained considerable popularity in the last two decades. Box girders are designed to function three-dimensionally. Overlapping functions of the deck plates, stringers, and floor beams are replaced by the ability of the box to resist bending, shear and torsional loading. It is this structural efficiency of box girders that leads to their relative economy in construction and usage as bridge members (1, 2).

As a result of recent box girder bridge failures during construction (3), research activities have been dramatically increased, particularly in Great Britain and in European countries (4, 5, 6, 7). Most of these research efforts are directed to nonlinear analysis of component parts of steel box girders. These studies complement the numerous available methods of analyzing box shapes. The concepts of design have been that of preventing buckling of the component parts and of analyzing box shapes with due consideration to warping and distortion of the cross section in the elastic range of material behavior.

The status of box girder design and analysis has been reviewed a number of times in recent years (1966-1974) (1, 2, 8). Of
the available procedures, the one which is based on the prismatic folded plate theory of Goldberg and Leve (9) offers an accurate method of analysis. This method considers box girders as a series of rectangular plates interconnected along longitudinal joints. The analysis was developed using elastic plate theory for loads normal to the plane of the plates and using two-dimensional plane stress theory for loads in the plane of the plate. The analysis is limited to straight prismatic box girders composed of isotropic plate elements with no interior diaphragms and with simple end conditions. Scordelis (10) and Johnston and Mattock (11, 12) have utilized this method in their studies of box girders. The inability of the analysis to account for effects of interior diaphragms and anisotropic plate elements such as transversely stiffened web plates prevents the application of the above method to large size steel box girders. Recently the method has been modified to take into account other than simple support conditions as well as continuous box girders (13).

Since a box girder is made up of thin plates, the thin-walled beam theories developed by Vlasov (14) and Dabrowski (15) have been used as the basis of the refined methods of analysis. Wright, Abdel-Samad, and Robinson (16, 17) extended Vlasov's theory to consider stiffened plate elements as well as to include the effects of interior diaphragms. Two methods were formulated. The "plate element" method uses matrix analysis procedures. It treats the structure as an assemblage of plate elements and utilizes a fourier series solution. The "Generalized Coordinate" method formulates equilibrium equations for the cross section and employs an initial
parameter solution method like that of Vlasov (14). This approach permits consideration of flexible interior diaphragms and arbitrary support conditions. A simpler version of this analysis to determine distortional stresses has been developed based on an analogy to the theory of beams on elastic foundations and is called the BEF analogy (16, 17, 18).

Another analytical technique which has been applied to box girders as well as many other structural problems is the finite element method. A brief review of recent developments and the state of the art on application of finite element method to box girders has been completed by Sisodiya and Ghali (19). The first application of the finite element method to the analysis of box girders was by Abu-Gazaleh and Scordelis (20) where they used six degrees of freedom at a nodal point, three for plate bending \((w, \theta_x, \theta_y)\) and three for in-plane behavior \((u, v, \theta_z)\). Multi-cell rectilinear box girders were solved by Sawko and Cope (21) who represented the cells by in-plane elements alone. William and Scordelis (22) developed a finite element program to analyze box girder bridges of constant depth and arbitrary plan geometry. Later Crisfield (23) developed a computer program for the analysis of multi-cell, rectilinear or skew box girder bridges. His analysis assumes symmetry about the middle horizontal plane of the bridge.

In addition to these and other regular finite element analyses, there are some modified versions. In the "Finite Segment Method" (20) the basic structural elements used are formed by dividing each web and flange into a finite number of transverse segments.
Compatibility and equilibrium conditions are satisfied at selected points along the four edges of each segment. The "Finite Strip Method (24) is similar to the Finite Segment Method but with longitudinal elements along the length of the girder.

Practically all the methods are confined to elastic analysis of box girders. Very limited studies on the load carrying capacity or ultimate strength of box girders have been made. Parr (25) in 1968 reported his work on the ultimate strength of box shapes having stocky component plates and subjected to flexural loads only. In 1972 Corrado (26, 27) completed testing of two model box girders to failure in bending and torsion, and formulated a method of estimating the ultimate strength on the basis of research results on plate girders. No mathematical or analytical procedure, however, was provided for the evaluation of stresses at the component parts when some parts of the box girder have been stressed beyond the elastic limit.

1.2 Objectives and Scope

The major objective of this work is to develop a procedure for the evaluation of the ultimate strength of steel box girders. The stress distribution and displacements of box girders in the elastic and inelastic range of behavior are also sought.

The finite element method is chosen for this study because of (1) the ability to include material nonlinearities for analysis in the inelastic range, (2) its ability to incorporate diaphragm stiffness for examining its effect, (3) its capability of handling different
isotropic and orthotropic components such as reinforced concrete decks and stiffened plates if desired.

In this study, complete load displacement relationship is investigated. To keep the extent of the study manageable in the non-linear analysis, buckling of the component parts is excluded, as is commonly done in stress evaluation of box girders. The method will over estimate the ultimate strength if buckling occurs in early stages of loading. Single cell, rectangular, prismatic and straight box girders are the object of the study because there are experimental results readily available on ultimate strength for comparison (26, 27, 28).

In the course of studying the ultimate strength, the behavior and stresses of the box girders in the elastic range are also examined. Results from tests and from other methods of solution are to be compared.
2. ELASTIC ANALYSIS OF BOX GIRDERS

2.1 Problem Formulation and Solution

The basic steps of the finite element theory and its application can be found in many references (29, 30, 31). The finite element analysis of an elastic continuum consist of (a) discretization of the continuum into a mesh of finite elements; (b) evaluation of the element properties; (c) assembly of the element properties into a global stiffness matrix and incorporation of the boundary conditions; (d) solution of the simultaneous equations. Only the necessary steps of the analysis pertinent to this study will be presented.

2.1.1 Discretization of the Continuum

The basic structural elements used in this analysis of straight box girders with rectangular cross section are rectangular in shape. The elements are formed by dividing transversely and longitudinally the webs and flanges as well as the diaphragms into an assemblage of small rectangular finite elements, Fig. 2.1.

The selection of element shapes strongly influences the simplicity of the problem formulation and solution. For box girders of non-rectangular shape and with curvature, triangular, quadrilateral and curved elements may need to be used. The subsequent evaluation and transformation of element properties from the element coordinate system into the global coordinate system require more work than that.
for rectangular elements. Since the structure under study is rectangular in shape and the primary objective is to obtain load displacement relationship in the inelastic range of material behavior which would require small sized elements, rectangular elements are chosen. This way the element properties can be evaluated in the global coordinate system with no need of transformation.

The size of the rectangular elements can be varied as desired throughout the structure. In regions where the anticipated stress gradient is high, such as locations near loading zone and supports, a fine mesh of elements can be used. The thickness and material properties of the elements can also be varied throughout the structure to accommodate different plate thickness and various materials.

2.1.2 Evaluation of Element Properties

Element properties are expressed as the stiffness matrices of the elements. There are mainly three methods for deriving the stiffness matrix of a finite element: the displacement method, the equilibrium method and the mixed method. In the first method, a displacement field is assumed within the element and the element stiffness matrix is derived from the minimum potential energy (29, 30).

For the equilibrium method, a stress field is assumed which satisfies the equations of equilibrium and the element stiffness matrix is derived from the principle of minimum complementary energy (30, 31, 32). The mixed method assumes both an equilibrium stress field and a displacement field separately within each element, the element stiffness matrix is derived from the variational principle (33, 34, 35).
In the displacement method, if the assumed displacement field is compatible, then the stiffness of the actual structure is always overestimated and monotonic convergence to the correct solution from below is ensured. Similarly, if the assumed stress field satisfies the equilibrium of forces at the boundary, then monotonic convergence to the correct solution occurs from above when the equilibrium method is employed.

In the literature, the displacement method is used extensively because it is relatively easy. The other two methods usually result in a greater number of total degrees of freedom and greater semi-band width of the stiffness matrix, thereby increasing the computational effort (30, 35). It was therefore decided to use the displacement method for this study, although in some cases the displacement method gives slightly less accurate results of stresses when compared with other methods.

The displacements \( \{f\} \) at any point within the element are approximated by shape functions \([N]\) associated with the generalized coordinates \(\{u\}\) which are the nodal point displacements.

\[
\{f\} = [N]\{u\} = \begin{bmatrix} N_1, N_j \ldots \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ \vdots \end{Bmatrix}
\]  
(2.1)

With the displacements known (or assumed) at all points within the element the strain \(\{\varepsilon\}\) at any point can be determined.

\[
\{\varepsilon\} = [B] \{u\}
\]  
(2.2)

In the equation \([B]\) is a matrix relating the nodal point displacements to the element strains. It is obtained through differentiating the
shape functions \([N]\) and rearranging them. It depends on the geometrical dimensions of the finite element and is related to the type of element and displacement field selected. Matrix \([B]\) is independent of material properties.

From the material constitutive law, the stresses at a point are given by

\[
\{\sigma\} = [D](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\}
\]

(2.3)

Where \([D]\) represents the elasticity matrix containing appropriate material properties, \(\{\epsilon_0\}\) is the initial strain vector, \(\{\sigma_0\}\) is the initial stress vector, and \(\{\sigma\}\) denotes the stresses within the element.

By applying to the element the virtual work principle or the theorem of minimum potential energy, element stiffness matrix \([k_e]\) is obtained as

\[
[k_e] = \int_{\text{volume}} [B]^T[D][B] \, d(\text{volume})
\]

(2.4)

In evaluating the element stiffness matrices for this study, it is assumed that the girder is made of thin walled members so that Kirchoff's assumption is valid: that plane sections normal to the middle surface of the plate remain plane after deformations. The inplane and bending displacements are assumed to be small in comparison to the dimensions of the box girder. This implies that the additional forces due to change of geometry are neglected.

**Plate Bending**

The plate bending behavior can be described by the out-of-plane displacement, \(w\), of the middle plane. Other parameters to
ensure at least an approximate satisfaction of slope continuity are rotations about the x-axis \((\theta_x)\) and rotations about the y-axis \((\theta_y)\). By using the sign convention shown in Fig. (2.1b), the displacement field which describes the bending deformations can be expressed in vector form as

\[
\{f\} = \begin{pmatrix}
w \\
\theta_x \\
\theta_y \\
\frac{\partial w}{\partial y} \\
\frac{\partial w}{\partial x}
\end{pmatrix}
\]  

(2.5)

Evaluation of the properties of elements which have the displacement "\(w\)" as the nodal parameters are described in Refs. 29 to 31 and 36 to 38. Generally, higher order elements give improved accuracy when few elements are used. These elements may better satisfy the boundary conditions and the displacement field may be closely approximated, but more time is usually required to generate the element stiffness matrices.

In this analysis the ACM (Adini, Clough and Melosh) plate bending element stiffness is used (29, 37). The ACM element is non-conforming in that slope continuity is not satisfied along the boundaries except at the nodal points. However as the number of elements is increased the solution converges to the correct value. Comparisons for plate bending elements are given in Refs. 36, 37 and 39.

In the ACM element a polynomial expression is used to define the displacement field "\(w\)" in terms of twelve parameters.
the element stiffness matrix is obtained using Eq. 2.4 and it is given explicitly in Refs. 29 and 37.

\[ w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} x y^3 \]  

(2.6)

In-Plane Behavior

The in-plane behavior of an element includes the displacements \( u \) in the \( x \)-direction and \( v \) in the \( y \)-direction, the normal strains \( \varepsilon_x \) and \( \varepsilon_y \) in these directions, and the shear strain \( \gamma_{xy} \). The evaluation of element properties for in-plane behavior are described in Refs. 29, 37 and 40.

In this analysis the "linear strain rectangle" element presented by Clough (42) is used for which the displacement polynomials are

\[ u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy \]  

\[ v(x,y) = a_5 + a_6 x + a_7 y + a_8 xy \]  

(2.7)

In the elastic range of material behavior, the elasticity matrix for in-plane displacement is given by

\[
[D^e] = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 0 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]  

(2.8)

or for more general cases, the compliance matrix is given by Eq. 2.9.
By using Eq. 2.4 and Eq. 2.9 the element stiffness matrix \([k_i]\) for in-plane behavior is derived and is given in Table 2.1. A similar but slightly different stiffness matrix has been presented in Ref. 42 resulting from the same displacement function. The difference is due to the condition that the element \(D_{13}\) of Eq. 2.9, which is nonzero in the nonlinear range of material behavior, was omitted in Ref. 42. This particular term is essential for the inelastic analysis in this study (43).

Superposition of In-Plane and Plate Bending Behavior

With the assumption that the displacements are small, the in-plane behavior and the out-of-plane behavior of an element are uncoupled. Thus the total stiffness matrix of an element \([k_e]\) can be obtained by direct combination of the in-plane stiffness matrix \([k_i]\) and the plate bending stiffness matrix \([k_b]\). In this analysis evaluation is made first for the in-plane stiffness then for the plate bending stiffness for the nodal points \(i, j, k, l\) sequentially (Fig. 2.1b) to obtain the total stiffness matrix for the element. Nodal point displacements are in the order of

\[
\mathbf{u}_i = \begin{bmatrix}
  u \\
  v \\
  w \\
  \theta_x \\
  \theta_y
\end{bmatrix} = \begin{bmatrix}
  u \\
  v \\
  w \\
  \frac{\partial w}{\partial y} \\
  \frac{\partial w}{\partial x}
\end{bmatrix}
\]

(2.10)
2.1.3 Assembly and Solution

The individual element stiffness matrixes \([k_e]\) are transformed from their local coordinate system into a global coordinate system \([k_e]_g\)

\[
[k_e]_g = [T]^T [k_e] [T]
\] (2.11)

where \([T]\) is the transformation matrix for nodal point forces from local coordinates into global coordinates (29).

The total global stiffness matrix \([K]\) of the box girder is obtained by summation.

\[
[K] = \sum_{e} [k_e]_g = \sum_{e} \int [T]^T [B]^T [D][B][T] \, d(vol) \quad (2.12)
\]

The total stiffness matrix relates the forces \([P]\) at the nodal points of the structure to the displacements \([\delta]\) of the nodal points (30).

\[
[K][\delta] = [P]
\] (2.13)

In this analysis the parallel nature of the local and global coordinate system is utilized (Fig. 2.1). Element stiffness matrices have been expressed directly in global coordinates rather than in local coordinates. This procedure omits the step indicated by Eq. (2.11).

From Eq. (2.10) each nodal point of an element will have five degrees of freedom. For the nodal point at the junction of two perpendicular planes such as the junction of a web and a flange, it will have six degrees of freedom. By adopting a proper sequence for numbering the nodal points such that the maximum difference in the nodal point numbers of the finite elements is minimized, advantage...
may be taken of the geometry of the structure to yield the minimum band width possible for the total stiffness matrix \([K]\). In the computer program (44), advantage is taken of the prismatic nature of the structure by numbering only the nodal points on the cross-section \(X = 0\). Nodal point numbers for all other specified sections along the length are automatically generated through computer programming.

The generalized nodal point forces \([P]\) include initial stresses and body forces. These forces can be applied through nodal points at any place on the structure. Distributed loading can be allocated to the nodal points using the consistent load vector concept (29).

Boundary conditions are handled through constraining the corresponding degrees of freedom. Any support condition can be closely approximated through this property of the finite element method. If other methods such as thin walled elastic beam theory or the analogy based on beam on elastic foundation are used, supports are assumed at the neutral axis of the girder. In a short span girder the effects of the support condition will be pronounced.

The Cholesky decomposition and backward substitution method is used to solve the large system of equations. This is the most time-consuming part of the problem in the computer.

2.2 Evaluation of Formulation

2.2.1 Two Degrees of Freedom versus Five Degrees of Freedom

As has been pointed out earlier each element nodal point has five degrees of freedom. The results of the finite element analysis
are the in-plane and out-of-plane displacements of these nodal points and the corresponding in-plane and plate-bending stresses of box girder component plates. In this analysis loads are assumed to be applied either through the diaphragms or at the junctions of the webs and flanges, and the out-of-plane displacements are considered relatively small. This implies that, for the structures and the loading conditions under study, a two degree of freedom condition instead of the five degree of freedom assumption may be employed. For a two degree of freedom case the band width of the total stiffness matrix, a very important factor in the solution time, is only about 1/3 of that for the five degree of freedom case with the same mesh divisions. Thus the solving of problems would be much more efficient using only two degrees of freedom if its use can be justified.

To investigate this, five problems are solved by using both two degrees and five degrees of freedom at a nodal point. Problem 1 to 4 use the same specimen, a single span rectangular composite box girder (Girder D1 of Ref. 28) with concentrated loads applied at the mid-span, Fig. 2.2. Problem 5 is a 100-inch long simply supported wide flange beam of W8 x 31 cross-section, Fig. 2.3. The details of the problems are summarized in Table 2.2.

In Table 2.2 the results are compared for maximum deflections, maximum stresses at the top and bottom flanges and the central processing time required by the CDC 6400 computer to solve the resulting simultaneous equations. It is obvious that, for these two structures, the two degree of freedom and five degree of freedom elements give practically identical results of stresses and
deflections. The maximum difference for all these example problems is less than one percent. The computational effort of the computer (CP time) for the five degree of freedom cases, however, is ten to fifteen times higher than that for the two degree of freedom system cases.

The number of operations in solving the simultaneous equations resulting from Eq. (2.13) is proportional to \(N^2\) where \(N\) is the total degree of freedom and \(B\) is the semi-band width of the total stiffness matrix. For a given structure and finite element discretization, the total number of degrees of freedom and the semi-band width both are approximately 2.5 times higher for the five degree of freedom system. This indicates that the solution time of the computer is about 15 times more for the former case. The number of computer input data cards, on the other hand, is exactly the same for both cases.

That the results from the two and five degree of freedom systems are practically identical for the structures of Figs. 2.2 and 2.3 is further indicated in Figs. 2.4 and 2.5. Figure 2.4 shows the deflection profile of the bottom flange to web junction of the box girder under bending and under torsion. Computed values have to be superimposed on each other in the figure along the entire half length of the box girder. In Fig. 2.5 the results from the two and five degree of freedom cases again fall on top of each other. Also shown in the figure is the stress distribution pattern computed by the simple beam (MY/I) theory in which the effect of shear on the stresses cannot be included. The influence of shear lag effect in the flanges is revealed by the results of the finite element analysis. This will be discussed later.
For both the box girder and the wide flange beam of Figs. 2.2 and 2.3 where the loads are applied at the edge of the plates, the plate bending stresses from the five degree of freedom system are small. This justifies the use of two degree of freedom formulation in this analysis. This condition will be demonstrated further in later examples.

2.2.2 Mesh Division Along the Length

Rectangular elements which have been widely used in two dimensional elasticity problems give mediocre results in beam analyses when the ratio of the element length to its width \((a/b)\) is greater than unity \((29)\). The deterioration in accuracy becomes more drastic when the aspect ratio \((a/b)\) increases from 1:1 to 4:1. This condition necessitates a finer mesh division along the length of the beam when finer mesh divisions are adopted for the cross-section.

The effect of the element aspect ratio on the accuracy of results is not as marked for box sections as for beams with rectangular cross-sections. This is because the longitudinal forces are mainly resisted by the top and bottom flanges where the rectangular elements replace the continuum better than the elements in the web do. To demonstrate this effect of the element aspect ratio, simply supported beams with three different cross-sectional shapes are analyzed. The cross-sections are a rectangle, a wide flange and a box as are shown in Fig. 2.6. The length of the beams is such that the resulting element aspect ratios vary from 1:1 to 4:1.

The results are summarized in Table 2.3, where the deflections and stresses are compared with known values from beam theory, including -18-
the effects of shear. The accuracy of using rectangular elements is poor for the rectangular beam even when the aspect ratio of 1:1 is used. For the wide flange and the box shapes, the accuracy for 1:1 elements is quite good, and for 2:1 elements it is acceptable, considering that a very coarse mesh is used for the analysis. In this example of beam bending without torsion, the wide flange and the box shape behave identically according to the beam theory. The differences in deflections and stresses are due to the fact that the box has double the web area of the wide flange beam. For steel box girders, the webs are usually slender and small in area relative to the flanges. The results of the finite element analysis using rectangular elements of a moderate aspect ratio can be expected to be better than those of Table 2.3.

To explore the effects of mesh division along the length and the element aspect ratios further, the box girder of Figs. 2.2a and 2.2b is analyzed using a different number of mesh divisions along its length. Both the simple bending and the pure torsional cases are investigated. The stresses and deflections at some points are plotted in exaggerated scale in Figs. 2.7 and 2.8 against the number of mesh divisions along the length of the beam. Also shown are the aspect ratio of the rectangular elements for the web. For all the stresses and deflections the computed values are "stabilized" when only a very few divisions along the half-length are used. For example, the maximum deflection under bending for 10 kip applied load, obtained by nine divisions along the half-length, is 0.04094 inch compared to 0.04123 inch by 30 division along the half-length. The difference is
only 0.67%. The computational effort, on the other hand, is more than five times higher in the latter case.

Therefore, in evaluating the behavior of box girders it is only necessary to have a moderate number of divisions along the length of the structure.

2.2.3 Mesh Division Across the Webs and Flanges

One of the advantages of the finite element method is that the shear lag effect is automatically included in the analysis. This has been indicated earlier, and further examination will be made later. Obviously the finer the element sizes across the cross-section, the more accurate the stress and deflection values are obtained.

How fine the elements should be is evaluated using different mesh divisions for the cross section of the box girder of Fig. 2.2 under simple bending and pure torsion. Ten divisions along the half-length is arbitrarily chosen. First the web is divided into three equal elements while the number of divisions across the flanges between the webs varies from one to eight. The results are tabulated in Table 2.4. For all simple bending cases, the computed deflections remain almost the same, changing from 0.03829 inch per 10 kip of load for one element across the flange to 0.03876 inch for the same load for eight elements across the flange. The computed flange stresses improves around 2%. The computational time of the computer, on the other hand, has increased 15 times. A similar situation exists for the pure torsional loading cases. The change in deflection estimate is very minor while the stress values improve 15 to 25%.

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The results of increasing mesh divisions across the web are also similar. Table 2.5 lists the deflections and stresses of the same box girder with two flange elements between the web and one to six web divisions. As the number of divisions increases, better results are obtained. The change (improvement) of results is at a higher rate than that for Table 2.4 where the number of flange mesh divisions is increased.

Usually mesh divisions across the webs and across the flanges are chosen such that the resulting elements have a moderate aspect ratio \((a/b)\). Therefore, fine elements in the web accompany fine elements in the flange. The computational effort for a fine mesh system often is fifteen to twenty times more than for coarse mesh divisions, while the accuracy of stress values increase by five to ten percent. Figure 2.9 depicts the relative increase of accuracy and computational time for the box girder under study. An approximate mesh division must be deduced from the importance of accuracy and availability of computer capacity and time. For the box girders under study, a flange mesh division of four with a web mesh division of three to four appears to give fairly good results without consuming too much computer time.
2.3 Comparison of Results with Existing Solutions

In order to check the validity of the assumptions and the accuracy of the analysis, results of a multicell box girder, a simple wide flange beam, a thin flange deck and a composite box girder are compared with the results obtained by other investigators.

2.3.1 Multicell Box Girder

A two span continuous bridge with a rectangular cross section of three cells is analyzed. The bridge has been analyzed by Scordelis using folded plate theory, the finite segment method and the finite element method (20). The dimensions and loading conditions are shown in Figs. 2.10a and 2.10b. The box girder is symmetric about the middle support thus a 60 ft. span cantilever with a simple support at the free end represents half of the bridge, Fig. 2.10c. Two different mesh divisions across the section have been used by Scordelis in his finite element analysis and are also adopted here, Fig. 2.10d. Along the length, eight divisions are used for a quarter of the bridge, Fig. 2.10c, whereas only seven divisions at different spacing have been employed in Ref. 20. Five degrees of freedom for each nodal point are used in this analysis.

The resulting vertical deflections along the top of the loaded web are shown in Fig. 2.11, together with the results of the folded plate theory and the two finite element models from Ref. 20. The vertical deflections of the bottom flange at cross sections 17.5 ft. and 30 ft. from the interior support are given in Fig. 2.12. From Fig. 2.11 it can be seen that the deflections along the loaded web
agree fairly well for all the methods considered. The results from this study are slightly higher, with a maximum difference of less than six percent of the folded plate theory values. The relative vertical displacement of the flange at the exterior webs are also slightly higher from this analysis, Fig. 2.12, but again the agreement among the results of all the methods is quite good.

There are a number of factors which contributed to the differences between the results of this analysis and those from the finite element method of Ref. 20. First the polynomials selected in the reference for the displacement field for out-of-plane deflection and for in-plane behavior are different from Eqs. (2.6) and (2.7), respectively. For in-plane behavior in addition to u and v, a third term for average rotation about the Z axis $\theta_{zi} = \frac{1}{2} \left[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \right]$ is considered in Scordelis' analysis. The six degree of freedom per nodal point system describes the deflections better than the five degree of freedom system used in this analysis. Secondly, the mesh divisions along the length are not the same. Nor are the loading conditions. In this study, the concentrated load is applied at a nodal point whereas in the reference the load is spread over two fine divisions of 6 inches each straddling the load point. Third, the support and diaphragm conditions are different. Complete fixity at the interior support and actual diaphragm thickness with a support restraint against vertical displacement of the bottom flange are the conditions of this analysis. The conditions that are used are not given in Ref. 20 for exact evaluation.
It must be pointed out that two degrees of freedom formulation of finite element analysis does not provide accurate results for this relatively thick walled concrete box girder. This is mainly due to the high plate bending rigidity of the plate elements which cannot be considered in a two degree of freedom formulation. In evaluating the plate bending rigidity of finite element, the elements in the compliance matrix contain terms to the third power of the plate thickness. Therefore only when the plate thickness is small, or when there is little out-of-plane bending, should the two degree of freedom system be used. To show how the thickness of a box girder affects the selection of two degrees of freedom and five degrees of freedom the same multicell box girder bridge of Fig. 2.10 is analyzed after it is transformed into an equivalent steel box girder. The cross section is transformed by using a ratio of elastic modulus $n = \frac{E_s}{E_c} = 10$. Two loading conditions are examined. In the first one, two 500 kip loads are applied symmetrically to the cross section to cause simple bending. In the second case two antisymmetric 500 kip loads formed pure torsional moment of 14,000 kip-ft. Mesh 1 of Fig. 2.10c is used with both the two degree and the five degree of freedom system. The resulting maximum deflections and stresses under the loading point are compared in Table 2.6.

For the concrete box girder, the difference in deflection between the results of the two systems is 13.8% for simple bending, which causes plate bending in the direction of the girder, and is 66.1% for pure torsional loading which generate plate bending in both longitudinal and transverse directions. When the cross section is
transformed into a steel section, the difference between the two and five degree systems is only 0.46% and 2.43% for simple bending and pure torsional loading respectively. For this reason, the multicell concrete box girder bridge is analyzed using five degrees of freedom per node, and the deflections are comparable to those obtained by Scordelis as shown in Figs. 2.11 and 2.12.

Comparing normal stresses, using the box girder bridge of Fig. 2.10 again, the values obtained from this analysis agree well with those obtained from Ref. 20. Figure 2.13 compares the normal stresses at the edge of the flange along the top of the loaded web. The maximum difference is at the load point where the element sizes and the manner of load application are different for the analyses considered as has been pointed out earlier. Between the interior support (the left end of Fig. 2.13) and the load point, the results are practically the same. Between the end support and the load point, the computed stresses are smaller than those of Ref. 20, definitely due to the rigidity of the end diaphragms and also possibly due to the support condition.

2.3.2 Wide Flange Beam

A simply supported wide flange beam (W8 x 31) of a 100-inch span is analyzed. The same beam has been studied to compare two and five degrees of freedom in Sect. 2.2.1. Due to symmetry with respect to mid-span, to the longitudinal axis and to the web, Fig. 2.3, only 1/8 of the beam need be considered in the finite element analysis. This implies that the load is applied vertically downward from the mid-height of the web and the supports are at the longitudinal axis of
the beam. This load and support condition make the results more comparable with the beam theory stresses and deflections. The Poisson's ratio is taken as zero to make it comparable with the beam theory. Because of the nature of the problem, two and five degree considerations give identical results, Table 2.2, thus the former is chosen. Ten divisions along the half span and six elements in the quarter of the cross section (three in the flange, three in the web) are used, adding up to 60 elements and 165 total degrees of freedom.

The computed deflections and the stresses along the centerline of the bottom flange are plotted in Fig. 2.14 together with the results from the beam theory. The beam theory deflections include the contributions of shear which is 7.5% at the centerline. Although the element aspect ratios are high (4:1) the deflections by the finite element analysis compare satisfactorily with those from the beam theory. The stresses from the two methods of analysis are practically identical. When 40 mesh divisions along the half of the beam length is used, the resulting deflections are practically the same. The maximum difference is at the centerline of the beam and is less than 1% from the results of Fig. 2.14.

2.3.3 Thin Flange Deck

One of the several tests performed by Schmidt (45) on plexiglas model flange decks is analyzed by the finite element method of this study. The deck, Fig. 2.15a is 0.405 cm (0.160 in.) thick, has a span of 157 cm (61.8 in.), has two inverted tees as webs and bottom flanges, and is loaded by two concentrated forces directly above the mid-span. Modulus of elasticity and the Poisson's ratio for the top
flange deck and the web are 33,700 kg/cm² and 0.384, and are 33,500 kg/cm² and 0.376 for the bottom flange respectively.

Because of the double symmetry, only one-quarter of the structure need be analyzed by the finite element procedure. Furthermore, because the bottom flange-to-web junction does not deflect laterally, simplification can be made in discretization. Two different mesh divisions across the cross section are used, being four and thirteen divisions of the top flange as shown in Figs. 2.15b and 2.15c. Along the length very fine meshes are adopted near the load and support points and relatively coarse ones in between, adding up to 21 divisions. The total number of elements, nodal point degrees of freedom, and corresponding semi-band width of the stiffness matrix are all given in Fig. 2.15.

The results of two degree of freedom formulation are compared with the test results and Schmidt's theoretical values. In Fig. 2.16 the normal stress in the top flange 1.0 cm (0.394 in.) away from the loading point are shown, non-dimensionalized using a beam theory bending stress at mid-span (45). The shear lag effect of the thin flange deck is very pronounced. Both the finite element analysis and Schmidt's procedure give good estimate of the normal stresses. The finite element method results are 3 to 6% lower than the test results. The displacements including the effects of shear are also obtained by the finite element analysis, but no deflections are given in Ref. 45 for comparisons.

To inspect the influence of plate bending rigidity, the structure is analyzed using four mesh divisions across the top
flange and the five degree of freedom formulation. Since the flanges and the webs are relatively slender (for example, the height to thickness ratio of the web is $8.99/0.304 = 23.6$) the results from two and five degree of freedom analyses are expected to be very close to each other. This is shown in Fig. 2.17 where the computed top flange normal stresses are compared. The maximum difference is only less than 1% at the cross-section, 1.0 cm away from the load and at the junction of the web and flange.

Also shown in the above figure are the results obtained from the thirteen mesh divisions of the top flange compared with the four mesh division stresses. The analysis provides almost the same results, indicating that few divisions are required to capture the behavior of the structure.

The example indicates that, for thin walled structures, a two degree of freedom formulation with moderately fine mesh divisions will provide fairly accurate results, including the effect of shear lag and shear deformations.

2.3.4 Composite Box Girder

The finite element analysis is applied to a composite box girder which is shown in Fig. 2.18. This box girder is specimen Dl of Ref. 28 and has been taken as an example in Sect. 2.2 for the evaluation of two or five degree of freedom analyses as well as for determining the importance of mesh divisions. It is one of the main structures to be analyzed later for the examination of inelastic behavior.
The box girder has a span length of 10 feet and an overhang of 2 feet at one end. It has one-sided transverse web stiffeners spaced almost equally throughout the entire length. The 0.076 cm (5/64 in.) thick web has a slenderness ratio of 158. The bottom flange thickness is 0.1875 cm (3/16 in.) so as to prevent buckling when loads are applied at the end of the overhanging part. Interior plate diaphragms of 3/16 in. thickness are located at the supports, at the mid-span and at the free end. Transverse loading stiffeners are also added at these points to prevent local failure under load. The concrete deck is 2.436 in. thick and is connected to the small steel flanges of the web with very closely spaced shear connectors to ensure complete interaction between the steel portion and the concrete deck. The modulus of elasticity and Poisson's ratio are 29,600 ksi and 0.3 respectively for the steel portion. The average concrete modulus of elasticity and Poisson's ratio are 3700 ksi and 0.15.

A two degree of freedom formulation is used in the analysis since it has been shown in Sect. 2.2.1 that this method generates almost the same results as the five degree of freedom system. Mesh divisions for the analysis are determined from the results of the Sect. 2.2.2 and 2.2.3. The web is divided into five divisions, the half flange is divided into two equal parts, and fourteen divisions are used for the half span length when the load is applied at the midpoint of the main span. The overhanging portion is disregarded for this loading condition and only half of the simple span needs to be analyzed. Spacing of divisions is close (2.5 in.) near the load point and the supports, and 5.0 in. between. When the load is applied at the
cantilever end, 26 divisions are used. Support conditions are made
as similar to the testing conditions as possible. In simply supported
cases, the vertical displacements are constrained along the support
line, and horizontal movement normal to the plane of the web is pre­
vented from one point. For cantilever loading, displacement of the
bottom flange along the length of the girder is also prevented in
addition to the above constraints. Actual thicknesses of the dia-
phragms are used in the analysis. Due to symmetry only one-half of
the cross-section is considered. The loads are separated into pure
torsion and pure bending and the resulting stresses and deflections
are superimposed to give the loading condition. By using two degrees
of freedom per nodal point, the stresses in the elements are assumed
to be constant throughout the thickness.

In the analysis the contributions of the transverse and the
longitudinal stiffeners are not considered. Similarly the orthotropic
properties of the reinforced concrete deck are also neglected. All
these can be incorporated into the analysis for more detailed results
if desired. Effect of changing the modulus of elasticity of the
concrete deck is examined in Sect. 2.4.

In Fig. 2.19 deflections under the web along the bottom flange
are compared. Shown in the figures are the results from the finite
element analysis using the two degrees of freedom per node, the test
results, and values by the thin walled elastic beam theory, including
shear deformations and warping rigidity of the girder. Figure 2.19a
is for loading at the mid-span with an eccentricity of 7.688 inches to
the vertical line of symmetry. The methods of analysis predict the
deflections rather closely. The maximum deflection by the finite
element analysis is 2.7% less than the test results. Similar results
are obtained for other magnitudes of eccentricity and load in Ref. 28.
In Fig. 2.19b the deflection profile under the web is shown when the
load is applied at the cantilever end with an eccentricity of 4.188
inches. The results from the thin walled elastic beam theory are
lower, partly because rigid diaphragms are assumed in the analysis.
The test results are slightly higher in the main span than the computed
values by the finite element procedure and slightly lower in the can-
tilever. The overall agreement is deemed quite satisfactory.

The normal stress distributions at a cross section (86.25 in.
from the left support) are shown in Fig. 2.20a for the same loading
condition as for Fig. 2.19a. Some test results are available and
these generally agree with the results of finite element and the thin
walled elastic beam theory analysis. The same general agreement among
the two methods of analysis and test results is also evident in Fig.
2.20b which shows the normal stresses due to a bending load only with-
out torsion.

In both Figs. 2.20a and 2.20b as well as many cross sections
inspected, the stresses from the finite elements method are lower than
the stresses by the thin walled elastic beam theory. Since these dif-
ferences occur not only under bending plus torsion but also under
simple bending, the cause can not be attributed to the influence of
warping torsion or torsional deformation. The mesh divisions of the
finite element analysis are sufficiently fine to avoid drastic inac-
curacy. The influence of two or five degree of freedom formulation
is only very small for this structure, as it is indicated in Sect. 2.2.1. It is possible that the cumulative of these factors result in the difference, but not probable. The main contributing factor appears to be the basic assumption of girder depth and "thin" wall.

For all cases of analysis so far in this study, the depth of a box girder or a beam has been taken as the centerline distance between the top and bottom flanges. This depth is larger than the actual height of the web. Consequently, the computed displacements and normal stresses are lower than those obtained by using the actual web depth. For example, the maximum vertical deflection for simple bending of the main span of the composite box girder is 0.003887 in. per kip of concentrated load at the mid-span if the actual web depth is used as the box girder depth. It is 0.004657 in. when the center-to-center distance between flanges is adopted. This represents a 20% difference with respect to the result from the actual web depth. The corresponding maximum normal stresses at the bottom flange are respectively 0.7710 and 0.6714 ksi having a difference of 13.5%. This difference is reduced if the flange thickness is smaller, as for the thin flange deck, Fig. 2.17, for which the difference amounts to 4%. In all cases, the more realistic magnitude of deflections and stresses lie between those two computed values and can be obtained only by using a special web element around that region. Such an approach is beyond the scope of this study. Reference can be made to some investigations on concrete bridges (37, 46, 47). Hereafter in this study the center-to-center distance between the flanges is adopted so as to be consistent with the thin walled elastic beam theory.
2.4 Examples of Applications

The procedure of analysis heretofore developed in this study aims at sufficiently accurate results and reasonably short duration of computer processing time. This capability serves as a basis for analysis of straight box girders in the inelastic range of material property. This also enables examination of box girder behavior in the elastic range as it is influenced by girder geometry and loading conditions. For example, the effects of diaphragm rigidity and diaphragm spacing on the stress distribution and on box girder deflection and deformation can be systematically examined under various combinations of bending and torsion. A study of these effects is being conducted. A number of simple examples are given below to illustrate the capability of the procedure.

2.4.1 Diaphragm Rigidity

From the thin-walled elastic beam theory and other procedures of analysis, it is known that deflections and stresses are reduced if a rigid diaphragm is provided at the load point. How rigidity of the diaphragm influences the maximum deflection and shear stress is depicted in Fig. 2.21. The box girder is that of Ref. 28 (Fig. 2.18) with torsional load applied at the mid-span. All three diaphragms are of the same thickness, varying from practically zero to 10.0 inches. As the thickness is increased from zero, both the web shearing stresses and particularly the maximum deflection reduce rapidly. Above a certain thickness of diaphragm both the deflections and stresses are not affected by further increase of thickness. For this box girder,
a 1/4 in. thick diaphragm can be regarded as rigid with respect to deflection.

The stresses in the diaphragm itself are influenced more by the loading and supporting conditions than by the diaphragm plate thickness. Within the elastic range of material property and without plate buckling, the stresses are almost inversely proportional to the plate thickness for a given loading and geometrical condition. Figure 2.22 shows the principal stresses at the 0.1875 in. thick diaphragm at the loading point of the box girder of Fig. 2.18. Fine mesh divisions may be used and lines of equal stress intensity can be plotted through interpolation, but this is not done here.

2.4.2 Diaphragm Spacing

There are few existing guidelines for the determination of diaphragm spacing. By using the same box girder of Fig. 2.18 both with different number of equally spaced diaphragms, some results are obtained from which qualitative conclusions can readily be drawn. The cases examined are: no diaphragm at all, diaphragms at the supports only, and additional diaphragms at 1/2, 1/3, 1/4, 1/5 and 1/6 of the span lengths, respectively. All diaphragms are 0.1875 in. thick which can be almost considered as rigid according to Fig. 2.21. Two different loading positions are investigated, one at mid-span the other at quarter points.

The vertical deflections at the bottom of the web along half span lengths are plotted in Fig. 2.23 for a typical torsional load applied at the mid-span. The deflections are almost the same when there
is no diaphragm at all, or when only support diaphragms are provided. These deflections are many times of these when intermediate diaphragms are used. Figure 2.24 shows an exaggerated scale of the deflection for the cases with intermediate diaphragms. When the applied torsion is between two diaphragms (case L/3 and L/5) the deflections and consequently rotations and distortion of cross-sectional shapes are relatively large only between these two diaphragms. If a load is applied at a diaphragm (cases L/2, L/4 and L/6) the deflections are practically the same regardless of diaphragm spacing.

This phenomenon is further illustrated by Figs. 2.25 and 2.26 which are the deflection profiles of the box girder under torsional loads at the quarter points. When there is no diaphragm between supports, the deflections are many times higher than those when intermediate diaphragms exist, Fig. 2.25. Whenever the load is between two intermediate diaphragms (case L/2, L/3, L/5 and L/6, Fig. 2.26) the deflections are relatively large only between the two adjacent diaphragms. Beyond these adjacent diaphragms, the deflections (and rotations and distortions) are practically the same as those when there is a diaphragm at the load point (case L/4). Obviously, the closer the diaphragms, the smaller the deflection between two diaphragms when load is applied therein.

The effect of diaphragm spacing on the stresses in the box girder follow the same pattern of deflections.

It must be pointed out that diaphragm spacing has little effect on bending of box girders. For example, the maximum deflection
of the box girder for a unit bending load at mid-span changes from 0.00386 in. with diaphragms at the ends to 0.00381 in. with diaphragms at one-sixth points of the span. The corresponding normal stresses at the bottom flange are 0.571 and 0.570 ksi, respectively.

2.4.3 Web Slenderness and Concrete Flange Rigidity

Further applications of the analysis are made to investigate the effects of component dimensions or rigidity on the deflection and stresses of the composite box girder which is being studied. The web thickness is taken as the variable, changing from 0.060 in. to 0.240 in. and making the web slenderness ratio 200 and 50, respectively. A simple bending load case and a pure torsional load case are investigated separately. Loads are applied at the mid-span of the simply supported box girder. Two divisions of half the flange between webs and three divisions of the web are selected with ten divisions along the half length of the girder for the analysis. The results of maximum deflection and stresses at some points at the mid-span are tabulated in Table 2.7. For bending loads, as the web thickness gets thinner and the web slenderness ratio increases from 50 to 200, the maximum deflection increases almost linearly. Both the top and bottom flange normal stresses increase, with the lower flange increasing more because the neutral axis shifts up for more slender webs. The web shearing stresses increase drastically for thinner webs, but the shear flow only changes moderately. Under torsional loading, the behavior is similar. The shearing stresses in the web increase fast with increasing slenderness ratio.
In the analysis of the composite box girder in all the examples so far, the modulus of elasticity of the concrete is taken as 3700 ksi. The effect of modulus of elasticity of the reinforced concrete flange deck is investigated by changing its value from 2500 ksi to 3000 ksi. The same mesh divisions and loading cases in the web slenderness examination are used. The maximum bottom flange deflection at the mid-span under a unit bending load decrease from 0.0420 in. to 0.0368 in. when the modulus of elasticity is increased from 2500 to 5000 ksi. This amounts to a 13% decrease of deflection for an increase of 100% in Young's modulus. The normal stresses in the top and bottom flanges increases from -0.273 ksi to -0.287 ksi and decrease from 5.811 to 5.688 ksi, a 5% and -2% change. Under pure torsional loading the percentage of changes are smaller. The deflection decreases 7%, the web shear decreases 1%, and the top flange shearing stresses decrease less than 1% for an increase from 2500 to 5000 ksi in the modulus of elasticity. These results indicate that the modulus of elasticity of the concrete is not a very important parameter in the analysis of composite box girders especially in torsion.
3. ULTIMATE STRENGTH

3.1 Nonlinear Behavior

The ultimate strength of box girders is associated with nonlinear behavior of the box girders. The nonlinearity is either caused by the nonlinear properties of the material or the nonlinear change of the box girder geometry when subjected to load.

When the deflections of the box girder, particularly the deflections of the component parts of the box girder, are large enough to significantly change the geometry of the structure, the equation of equilibrium must be formulated for the deformed configuration. The strains in the displacement equations may include higher order terms for the large deflection analysis. The instability of the structure or its components can be formulated as an eigenvalue problem. Numerous studies have been carried out to investigate the geometrically nonlinear behavior and stress distribution of the box girder flanges and webs (4, 5, 48, 49).

Material nonlinearity of box girders is due to nonlinearly elastic and plastic or viscoelastic characteristics of the structural material. Infinitesimal linear strain-displacement approximations usually form an adequate basis for the evaluation of the displacements and stresses under the condition of material nonlinearity. However, the complexity arises from plastic behavior under multiaxial
stress states. This fact together with the history dependence of strains at different points of a structure makes even the solution of a simple problem a formidable task. Consequently, existing solutions to flange and web behavior due to material nonlinearity involve gross simplifications. On the other hand, it has been shown that the incremental step procedure for nonlinear displacement analysis is ideally suited for structures with inelastic material properties (50, 51).

To evaluate the ultimate strength of box girders, both the geometric and material nonlinearity should be considered. For case studies which analyze the behavior of a specific box girder from the linear elastic range to failure, such thorough and complex examinations may be carried out. For more practical estimates of nonlinear behavior and load carrying capacity of steel box girders as a basis of design, judgment must be made with regard to simplifications. On the assumption that the instability of box girder components are prevented and that the deflections are not excessive, attention in this study is directed to the investigation of the nonlinear behavior of box girders caused by the nonlinear properties of the material alone.

Rectangular box girders are the objects of the study. Loads on the girders are applied at junctions of component plates. The finite element method is used for obtaining complete load-displacement relationship considering material nonlinearity and by employing the incremental approach.
The essentials of finite element analysis in this study have been presented in Chapter 2. It has been pointed out that for small out-of-plane bending of component plates, the formulation of the stiffness matrix by using two degrees of freedom per nodal point provides sufficiently accurate results with only a moderate amount of computational effort. Two degree of freedom formulation is adopted for the nonlinear analysis.

3.2 Material Properties

In the elastic analysis, Chapter 2, the relationship between stress and strain has been considered linear. The elasticity matrix $[D]$ of Eq. 2.3, thus can be readily evaluated. For the evaluation of the nonlinear behavior of box girders due to nonlinear characteristics of the material, these characteristics must first be defined. In this study of steel and composite box girders, since normal stresses transverse to the plane of the girder component plates are neglected, only plane stress constitutive rules need to be described.

3.2.1 Stress-Strain Relationship

A linear elastic, limited plastic flow, and linear strain hardening stress-strain relationship in uniaxial tension and compression is assumed for the steel, Fig. 3.1a. In the linear elastic range, the steel elements of the finite element mesh obey Hooke's Law $(E)$ and Poisson's ratio $(\nu)$. The yield stress level $(\sigma_y)$ and the strain hardening modulus $(E_{st})$ are determined by actual uniaxial tension testing. In the analysis, any stress-strain curve different
from the one defined here can be accommodated through either three linear lines or the adoption of the Ramberg-Osgood equation.

Reinforced concrete is a heterogeneous material, but it is considered homogeneous in a macroscopic sense. Furthermore, it is assumed to be isotropic in this analysis, although its orthotropic nature can be handled in the finite element analysis. The uniaxial tension-compression stress-strain properties are shown in Fig. 3.1b. Under tension and compression, the stress-strain relationship is assumed linear with a slope of $E_c$. This assumption enables the utilization of the results of linear elastic analyses from Chapter 2 as the basis for nonlinear evaluation.

The concrete is assumed to crack at a tension stress 15% of the ultimate compressive stress, $f'_c$; $f'_t = 0.15 f'_c$. Under compression the proportional limit is taken as $0.70 f'_c$. Between the proportional limit and $f'_c$ at a strain of 0.002 in. per in., the stress-strain curve is defined by a Ramberg-Osgood equation (46)

$$
\varepsilon = \frac{\sigma}{E_c} + \left(1 - \frac{m}{n}\right)\left(\frac{f'_c}{E_c}\right)^n \left(\frac{\sigma}{f'_c}\right)^n
$$

(3.1)

where $\sigma$ and $\varepsilon$ are the corresponding stress and strain on the curve, $E_c$ is the modulus of elasticity of the linear portion, and $m$ and $n$ are constants (46). Beyond $\varepsilon = 0.002$, to the crushing strain of $\varepsilon_f = 0.005$, a horizontal straight line (zero slope) is adopted. By Ref. 46, the values of the constants are taken as $m = f'_c / 0.002 E_c)$ and $n = 9$. Any of these values can be adjusted according to actual results of testing.
3.2.2 Yield Criteria

The elements are subjected to biaxial states of stresses, \( \sigma_x \), \( \sigma_y \) and \( \tau_{xy} \). The initial yield criterion for plane stress problems is represented by a relationship of the form

\[
F(\sigma_{ij}) = K
\]  

(3.2)

where "\( F \)" is generally referred to as the loading function and \( K \) is the yield condition, a known predetermined material constant. \( \sigma_{ij} \) represent the stress tensor. Equation 3.2 represents a closed, convex curve in a two-dimensional space (52). It is the initial yield curve for steel or the failure envelope for concrete.

In this analysis Von-Mises yield criterion (52) is used for steel. This criterion can be expressed as

\[
F(\sigma_{ij}) = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} = \sigma_o
\]  

(3.3)

where \( \sigma_o \) is the uniaxial yield strength, or in terms of principal stresses

\[
F(\sigma_{ij}) = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2} = \sigma_o
\]  

(3.4)

An effective stress \( \sigma_e \) may be introduced and is defined as

\[
\sigma_e = \sqrt{\sigma_x^2 + \sigma_y^2 + 3\tau_{xy}^2 - \sigma_x \sigma_y}
\]  

(3.5)

By combining Equations 3.3 and 3.5, Equation 3.3 can be written simply as

\[
\sigma_e - \sigma_o = 0
\]  

(3.6)
Initial yielding occurs when the effective stress $\sigma_e$ equals the uniaxial yield stress of the material, $\sigma_0$. The von-Mises initial yield curve is plotted in Fig. 3.2a.

The failure envelope of concrete under biaxial stresses has been investigated by numerous researchers (43, 54, 54). Essentially it can be simplified as in Fig. 3.2b. Failure can be either by cracking (portion ABC of Fig. 3.2b) or crushing (portion ADC). Experiments show that the strength of concrete under biaxial tension is almost independent of the principal stress ratio $\sigma_1/\sigma_2$ and is equal to the uniaxial tensile strength, $f'_t$. Under biaxial compression, for simplicity in this study, the von-Mises yield criterion with $\sigma_0 = f'_c$ is employed to approximate the failure curve. For tension-compression stress states, straight lines connecting $f'_t$ and $f'_c$ are adopted except for the portions $0 \leq \left( \frac{\sigma_2}{\sigma_1} \text{ or } \frac{\sigma_1}{\sigma_2} \right) \leq \frac{1}{15}$. In this portion the concrete does not crack and the crushing failure curve is extended into the regions.

3.2.3 Subsequent Yielding and Flow Rule

In Equation 3.2, it is defined that whenever the function $F$ becomes equal to the constant $K$, yielding will begin. After stress has reached yielding at a point, subsequent behavior will depend on the strain hardening characteristics of the material as in the case of uniaxial stress state. For a perfectly plastic material, the initial yield curve remains fixed. For a material that strain hardens, the yield curve must change for continued straining beyond the initial yield. Equation 3.2 can be generalized to describe subsequent yield curves. After yielding has occurred, $K$ takes on a new value depending
on the strain hardening characteristics of the material. There are three different cases for a strain hardening material:

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} \sigma_{ij} > 0 \quad \text{Loading} \]  
\[ (3.7a) \]

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} \sigma_{ij} = 0 \quad \text{Neutral Loading} \]  
\[ (3.7b) \]

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} \sigma_{ij} < 0 \quad \text{Unloading} \]  
\[ (3.7c) \]

If the material is unloaded and loaded again, additional yielding will not occur until the new value of \( K \) is reached.

In this study, a work hardening material will be assumed with an isotropic strain hardening \((52)\). The value of \( K \) for the subsequent yielding in the incremental theory of plasticity can be calculated from the amount of the plastic work or from equivalent plastic strains.

Then the yield function, \( F \), can be derived as a function of the equivalent plastic strain, \( \varepsilon_p \), \((52, 55)\).

\[ F (\sigma_{ij}) = H (\varepsilon_p) \]  
\[ (3.8) \]

or using the von-Mises yield criterion, Eq. 3.6,

\[ \sigma_e - H (\varepsilon_p) = 0 \]  
\[ (3.9) \]

In the analysis the experimental uniaxial stress-strain relationship is used for the evaluation of the functional relationship.

So far the yield condition and the loading functions have been defined. Because of the dependence of the plastic strains on the loading path, it becomes necessary to compute the increments of plastic strains throughout the loading history. In order to express
the increments of plastic strain on the basis of the present state of
stress and the stress increment, another assumption has to be made.

The associated flow rule of the von-Mises yield criterion
which is the "Prandtl-Reuss Equation" is assumed to be valid for the
steel, as well as for the concrete after crushing in compression. This
principle can be expressed as

\[ \dot{\varepsilon}^P = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \]  

(3.10)

where \( \dot{\varepsilon}^P \) is the plastic strain increment and \( d\lambda \) is a proportional
constant. This constant defined for the case of von-Mises yield
criterion is

\[ d\lambda = \frac{3}{2} \frac{\dot{\varepsilon}_e}{H'} \]  

(3.11)

where \( \dot{\varepsilon}_e \) is the incremental effective stress and \( H' \) is the slope of
the effective stress versus strain curve.

Usually, the incremental stress \( \dot{\sigma} \) and the incremental strain
\( \dot{\varepsilon} \) are directly related by an elasticity matrix. In the elastic
range,

\[ \{\dot{\sigma}\} = [D] \{\dot{\varepsilon}\} \]  

(3.12)

The elasticity matrix \([D]\) is constant and is given by Equation 2.8.
In the inelastic range, the compliance matrix is called \([D^eP]\). The
definition of \([D^eP]\) is based on

\[ \dot{\sigma} = [D^eP] \{\dot{\varepsilon}\} \]  

(3.13)

a derivation similar to that used by Yamada (56, 57) and is shown
below.
The total strain increment \( \{ \dot{\varepsilon} \} \) can be written as the summation of elastic strain increment \( \{ \dot{\varepsilon} \}^e \) and the plastic strain increment \( \{ \dot{\varepsilon} \}^p \).

\[
\{ \dot{\varepsilon} \} = \{ \dot{\varepsilon} \}^e + \{ \dot{\varepsilon} \}^p
\]  

(3.14)

The elastic strain increment has been defined previously

\[
\{ \dot{\varepsilon} \}^e = [D^e]^{-1} \{ \dot{\sigma} \}
\]  

(3.15)

For an isotropic material if \( \{ \sigma \} \) represent a stress state of an element just before it undergoes the additional strain \( \{ \dot{\varepsilon} \} \), the deviatoric stress state \( \{ \sigma' \} \) corresponding to \( \{ \sigma \} \) is

\[
\{ \sigma' \} = \begin{cases}
\sigma'_x \\
\sigma'_y \\
\tau_{xy}
\end{cases} = \begin{cases}
\sigma_x - \frac{1}{3} (\sigma_x + \sigma_y) \\
\sigma_y - \frac{1}{3} (\sigma_x + \sigma_y) \\
\tau_{xy}
\end{cases}
\]  

(3.16)

For the von-Mises yield criterion, \( \{ \sigma' \} = \frac{\sigma^f}{\sigma_{ij}} \). Then Equation 3.10 can be written as

\[
\{ \dot{\varepsilon} \}^p = \{ \sigma' \} \, d\lambda
\]  

(3.17)

From Equations 3.14, 3.15 and 3.17

\[
\{ \dot{\varepsilon} \} = [D^e] \{ \dot{\sigma} \} + \{ \sigma' \} \, d\lambda
\]  

(3.18)

or

\[
\{ \dot{\sigma} \} = [D^e] [\dot{\varepsilon}] - [D^e]^{-1} \{ \sigma' \} \, d\lambda
\]  

(3.19)

The effective stress \( \sigma_e \) can be expressed in terms of the deviatoric stresses as

\[
\sigma_e^2 = 3 \left( \sigma_x'^2 + \sigma_y'^2 + \sigma_x' \sigma_y' + \tau_{xy}'^2 \right)
\]  

(3.20)
\[
2 \sigma_e \dot{\sigma}_e = 3 \left[ \sigma_x \left( 2 \dot{\sigma}_x + \dot{\sigma}_y \right) + \sigma_y \left( 2 \dot{\sigma}_y + \dot{\sigma}_x \right) + 2 \tau_{xy} \right] (3.21)
\]

From Equation 3.16
\[
2 \dot{\sigma}_x + \dot{\sigma}_y = \sigma_x
\]
\[
2 \dot{\sigma}_y + \dot{\sigma}_x = \sigma_y
\]
\[
\tau_{xy} = \tau_{xy}
\]
which, when inserted into Equation 3.21 results in
\[
2 \sigma_e \dot{\sigma}_e = 3 \left[ \sigma \right] \left[ \sigma \right]^T (3.23)
\]
Equation 3.11 can then be written as
\[
dl = \frac{9}{4 \sigma_c^2} \left[ \sigma \right]^T \left[ \sigma \right] (3.24)
\]
By premultiplying Equation 3.19 by \( \left[ \sigma \right]^T \) and using the vector \( \left[ S \right] \) to represent \( [D^e] \left[ \sigma \right]^T \), Equation 3.24 becomes
\[
4 \sigma_e^2 H \frac{1}{9} d\lambda = \left[ S \right]^T \left[ \varepsilon \right] - \left[ \sigma \right]^T \left[ S \right] d\lambda (3.25)
\]
Defining
\[
S^* = \frac{4}{S} \sigma_e^2 H + \left[ \sigma \right]^T \left[ S \right] (3.26)
\]
then
\[
dl = \frac{1}{S^*} \left[ S \right]^T \varepsilon (3.27)
\]
By substituting this back into Equation 3.19
\[
\left[ \sigma \right] = \left[ D^e \right] - \frac{1}{S^*} \left[ S \right] \left[ S \right]^T \left[ \varepsilon \right] (3.28)
\]
Define

\[ [D^e_{\text{ep}}] = [D^e] - \frac{1}{S^*} [S] [S]^T \{\epsilon\} \quad (3.29) \]

Then \( \{\dot{\epsilon}\} = [D^e_{\text{ep}}] \{\dot{\epsilon}\} \) is as expressed in Eq. 3.13.

In cartesian coordinates, Eq. 3.28 can be expressed explicitly as

\[
\begin{bmatrix}
\frac{\partial \sigma_x}{\partial x} \\
\frac{\partial \sigma_y}{\partial y} \\
\frac{\partial \tau_{xy}}{\partial y}
\end{bmatrix} =
\begin{bmatrix}
\frac{E}{1 - \nu} \frac{S_2}{S^*} - \frac{S_1}{S^*} \\
\frac{\nu E}{1 - \nu} \frac{S_1 S_2}{S^*} - \frac{E}{1 - \nu} \frac{S_2}{S^*} \\
- \frac{S_1 S_3}{S^*} - \frac{S_2 S_3}{S^*} \frac{E}{2(1 - \nu)} - \frac{S_3^2}{S^*}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \varepsilon_x}{\partial x} \\
\frac{\partial \varepsilon_y}{\partial y} \\
\frac{\partial \gamma_{xy}}{\partial y}
\end{bmatrix}
\]

where

\[ S_1 = \frac{E}{1 - \nu} (\sigma_x' + \nu \sigma_y') \]
\[ S_2 = \frac{E}{1 - \nu} (\sigma_y' + \sigma_x') \]
\[ S_3 = \frac{E}{1 - \nu} \tau_{xy}' \]

The definition of \( S^* \) requires the value \( H' \) which is the slope of the effective stress versus effective strain curve. This slope may be obtained from the uniaxial stress-strain curve.
3.3 Solution Technique

3.3.1 Incremental Method

In the finite element solution of linear elastic problems Chapter 2, the answer of displacements are obtained by solving the simultaneous Equations 2.13. No trial and error or iteration is required since the stress-strain displacement relationship is linear. If the stress-strain relationship is nonlinear, solutions can not be obtained directly. Either an iterative method for certain classes of problems (29, 58, 59, 60) or an incremental procedure (30, 61, 62), or their combination i.e. an incremental iterative method (46, 62, 63) must be used.

In the iterative procedure, the total load is applied to the structural system. The stiffness matrix is derived based on the characteristics of one point on the stress-strain curve of the material. Thus equilibrium is not necessarily satisfied and iterations must be performed. This is done through successive adjustments of the elastic constants (E and ν) in the elasticity matrix [D] or [DE]. Such adjustments can be handled fairly easily for problems of nonlinear elastic material. For materials exhibiting plasticity, the compliance matrix [DEP], Eq. 3.29, is fully populated and the adjustments of the material property constants are extremely difficult if not impossible. Consequently, the iterative procedure can not be used here.

The basis for the incremental method is applying small increments of load successively and considering the structural behavior
as linear within each increment of load, Fig. 3.3. The total stress and strain under a given total load are obtained by summation.

For each increment of load \( \Delta P_i \) the relationship between this load increment and the increments in displacements \( \Delta \delta_i \) can be expressed as

\[
[K_1][\Delta \delta_i] = \{\Delta P_i\}
\]

for nonlinearity due to material properties only (64, 65). \([K_1]\) is the assembled total stiffness matrix for the load increment. If the initial load and displacement are \( \{P_o\} \) and \( \{\delta_i\} \) respectively, the total displacement and forces at any stage are given by

\[
\{P_i\} = \{P_o\} + \sum \{\Delta P_i\}
\]

\[
\{\delta_i\} = \{\delta_o\} + \sum \{\Delta \delta_i\}
\]

The procedure is the most general technique available for solution of elasto-plastic problems and is used particularly with the flow theory of plasticity. The incremental procedure is comparable to the Euler method of solving differential equations of initial value problems. Therefore, methods such as the Runge-Kutta procedure (66) or other procedures for the acceleration of the solution of initial value problems are also applicable.

3.3.2 The Tangent Stiffness Method

The stiffness matrix \([K_1]\) of Eq. 3.31 can be derived by using the tangent modulus, the secant modulus, or the initial modulus for the compliance matrix and the element stiffness matrix (29, 30, 46, 63). When the tangent modulus is used with the incremental procedure,
the method is called the incremental tangent stiffness method (57, 61, 62).

The method was initially suggested by Pope (61) for the solution of elastic-plastic problems by finite elements. It was then adopted by Marcal and King (62). Pope used an iterative procedure to determine the size of a load increment which initiates yielding at an additional element. Yamada (57) developed a closed form solution for the determination of the load increment. In this study, a procedure similar to that by Yamada is used. The size of the load increment \[ \Delta P_i \] at every stage is determined as a result of the analysis itself such that the load increment is sufficient to cause yielding in a specified number of elements.

The procedure used in this analysis is summarized as follows:

1. Apply an initial load vector, analyze the system elastically as described in Chapter 2, calculate displacements and stresses, calculate effective stress \( \sigma_e \) for each element.

2. Scale all the elastic values by a factor in order to induce first yielding at the element of maximum equivalent stress. Any additional load will cause inelastic behavior of the structure.

3. Calculate according to the stress-strain curve the elastic-plastic compliance matrix \( [D^{ep}] \) and henceforth the element stiffness matrix \( [k_p] \) for the post-yield elements.
4. Assemble the global stiffness matrix \([K_i]\), apply a dummy unit load vector, solve for the unknown displacements and evaluate the corresponding stress data.

5. Calculate the scaling factor \((r)\) for each element corresponding to its state of stress (elastic or inelastic) in order for the element to reach yielding, cracking, or ultimate strength.

6. Select a scaling factor \((r_m)\) such that a desired number of elements will be yielded at that increment of load, scale the stresses and strains with this factor and obtain the total stresses, strains, displacements and loads.

7. When the slope of the load deflection curve is less than a specified value, or if the load has reached a required value, the procedure is completed. Otherwise, return to step 3, and repeat.

This procedure requires more computational effort than needed for a load increment in the initial stiffness approach, but the tangent stiffness method permits a significant increase in the size of the load increment as compared to other methods.
3.4 Illustrative Problems

3.4.1 Wide Flange Beam

The solution technique is first applied to a fix-ended beam which is 14 feet long with a W8 X 40 cross section, and is loaded at the third points. It is one of the beams examined by Knudsen, Yang, Johnston and Beedle (67) for the analysis of inelastic behavior of beams. The beam is suitable for testing the procedure of this study because the effects of geometric nonlinearity are minimal until just before failure of the beam. This problem is also analyzed in Ref. 46.

The actual dimensions of the beam are as follows: flange width and thickness, 8.06 in. x 0.552 in.; web thickness, 0.370 in. and overall depth, 8.32 in. The modulus of elasticity and Poisson's ratio are assumed as 29,600 ksi and 0.3 respectively. The stress-strain curve is from testing a standard tensile specimen and shows a trilinear relationship. The yield point is 37.8 ksi. Onset of strain hardening occurs at a strain of 0.017 in./in. Tensile strength is 52 ksi with an average strain hardening modulus of 630 ksi (67).

The loading and the mesh size for the finite element analysis are shown in Fig. 3.4. Due to symmetry along the span, only half of the beam length is considered. Fifteen divisions are used along the half span. The loads are assumed to be applied at the center of the cross section so as to take advantage of the double symmetry of the cross section to cut down the total number of degrees of freedom and the band width in the analysis. The half flange is divided into two elements and the half depth into five. The center to center distance
between the flanges is considered as the depth. For this condition of mesh division, there are 272 simultaneous equations in Eq. 3.31 and the semiband width is 22. Other mesh divisions have also been examined resulting in different number of equations and bandwidths. The final result of ultimate strength or load carrying capacity of the beam, however, differs very little.

The load-deflection curve for the midspan, as obtained from this analysis, is shown in Fig. 3.5, together with the test data from Ref. 67. The curve can be approximated by three straight lines. The first linear part corresponds to the elastic behavior of the beam when stresses, strains and deflections are all proportional. The second straight line portion starts when yielding commences at the flange at the supports. As higher loads are applied, yielding penetrates from the flange into the web while strain hardening starts at the extreme fibers. When a very small load increment causes a very large increase in deflection, yielding at the supports has practically reached the neutral axis and the plastic hinges have formed. In the analysis, at this stage, 57 of the 105 elements have plastified and 14 have reached the tensile strength of the material. The beam has attained its ultimate strength or load carrying capacity.

The test results agree quite well with the predicted values from the finite element analysis. The test results are slightly lower due to the existence of residual stresses in the test beam, to the condition that the end supports are not 100% fixed, and to the inherent characteristic of the displacement method in overestimating the stiffness of a structure thus giving higher predicted strength.
The residual stresses, if known, can be taken into consideration in the analysis. Without considering residual stresses, the simple plastic theory predicts that first yielding occurs at 71.6 kips with a midspan deflection of 0.30 in. The corresponding values by this analysis are 65.9 kips and 0.313 in. being closer to the test results.

The effects of shear are not included in the simple plastic theory but are automatically considered in the finite element analysis. The load carrying capacity is 115.8 kips, 114.1 kips and 107.0 kips by testing, this study, and Ref. 67 respectively.

The most predominant factor for an accurate determination of the load-deflection behavior of the beam is the size of the load increment. Logically, if one element is allowed to yield at one increment of load, the most accurate result will be obtained for the mesh division chosen. This will also enable tracing the sequence of yielding of elements in the correct order. However, such a procedure requires many increments of load to determine the ultimate strength, a matter of computational effort. To speed up the computation, the load increments can be determined such that a prescribed number of elements will reach or exceed the yield stress during a load increment. As the load carrying capacity is approached, this method may not be practical since very large deflections accompany yielding of even one additional element. A second method of increment is imposed in terms of the maximum and minimum percentage of the total load. Higher percentages of load increments can be allowed at the beginning of yielding, and the percentages should be reduced as the rate of deflection increases for an increment of load. For this illustrative
example, the maximum percentage changes from 3% at first yielding to 0.4% at forming of the plastic hinges.

This illustrative problem demonstrates the workability and accuracy of the procedure for the elasto-plastic analysis to estimate the ultimate strength of structures with negligible geometric non-linearity.

3.4.2 Steel Box Girders

There are only very limited results from ultimate strength testing or analysis of box girders. The two small model specimens reported in Refs. 26 and 27 have been designed to observe the influence of web yielding on the load carrying capacity of the box girders. The webs failed by tension field action in resisting high shear which generated relatively large out-of-plane deflection of the web plates. The effects of geometric nonlinearity were therefore prominent, in addition to the effects of the material nonlinearity. For lack of test data, one of these model box girders is analyzed in this study for further evaluation of the procedure of analysis.

The dimensions of the model box girder are indicated in Fig. 3.6. The load is applied directly above one web at the mid-length, symmetrical to the beam span. The yield stress of the top flange, the webs, and the bottom flange are 32.5 ksi, 30.4 ksi, and 31.3 ksi, respectively. The corresponding tensile strength are 47.4 ksi, 43.4 ksi, and 45.6 ksi. For all component plates, the onset of strain hardening is assumed to start at a strain of 0.012 in./in. and to have
a strain hardening modulus of 500 ksi. The modulus of elasticity and the Poisson's ratio are 29,600 ksi and 0.3 respectively.

In the finite elements analysis, the transverse stiffeners are not considered. Furthermore, for symmetry, the intermediate X-diaphragm at the left half is ignored. The half span length is divided into eight parts, with closer mesh divisions (1 in.) at the support and loading point and relatively coarse ones (2 in.) in between. The flanges between the webs are each divided into two equal parts, and the webs into three. The cross-shaped diaphragms are approximated by plates having the same thickness as the webs and are connected to the box girder only at the four corners of the box. In the testing of the model, the midspan X-diaphragm did not appear to be sufficiently strong and failed before the attainment of the girder ultimate load. Consequently two conditions are considered in this analysis, one with the midspan diaphragm and one without.

The resulting load-deflection curves and the test results are shown in Fig. 3.7. As anticipated, the case of no loading diaphragm has higher deflections even under relatively low loads of 800 and 1000 lbs. The midspan X-diaphragm appears to be effective up to at least 1200 lbs. At 1400 lbs, the first definite sign of tangent diagonal web bulging appeared in panel 6 at the loaded side (26, 27). The effects of geometric change started to influence the load-carrying capability and the result of this analysis considering the midspan diaphragm underestimates the deflection. At 1600 lbs. the X-diaphragm has failure so that the lower curve in Fig. 3.7 is valid for comparison with test results. Again, the measured deflection is higher than
predicted because of the large deflections of the web plates. Actual failure occurred at 1800 lbs. due to local failure of the transverse stiffener at the load point. The computed load-deflection curve has an upward turn around this load magnitude because of the strain-hardening effect. By taking into consideration the failure of the X-diaphragm but not the tearing of the transverse stiffener, Corrado predicted a ultimate strength of 1700 lbs. whereas the finite element approach estimated 1894 lbs.

The influence of the size of load increment, as pointed out earlier, is apparently depicted in Fig. 3.7 by the upper curves. For the same mesh division and the same specified number of elements to be yielded at each load increment, the coarse load increments result in a higher load-deflection curve. If, for example, four elements are permitted to yield after an increment of load \( \Delta P_{i+1} \) from load \( P_i \), \( \Delta P_{i+1} \) is determined by a scale factor \( v_m P_i = P_i + \Delta P_{i+1} \) such that the fourth element will just reach yielding. The other three elements will have stresses higher than yielding. This condition of stress will be corrected through the total stiffness matrix of the structure, but only during the next increment of load. The resulting load-deflection curve is thus higher than the actual one. Therefore, when permissible, finer load increments should be used, and a small percent of load increase should be specified.

The example indicates that the procedure can be employed for ultimate strength evaluation of box girders with acceptable results, even when geometric nonlinearity exerts strong influence on the behavior of the box girder. An analysis of the other small model box
girder of Refs. 26 and 27 gives similar results. In fact, the computed stresses in the box girders agreed quite well with the measured values. Figure 3.8 shows the load versus shearing stress relationship of a point in the web of specimen M2 of Refs. 26 and 27. Before bulging of the web, the computed and measured shearing stresses are practically the same. Even after the web deflections start to increase, the predicted stresses are still acceptable.

3.4.3 Composite Box Girders

Two composite box girders, D1 and D2, have been subjected to bending and torsion and have been tested to failure (28). The behavior of girder D1 in the elastic range is examined in detail in Sect. 2.3.4. Its ultimate strength and behavior in the inelastic range are analyzed here using the procedure of this study. The geometry of the box girder is shown in Fig. 2.20. The web slenderness ratio and the stiffener spacing have been designed such that web buckling would not occur. Consequently the effect of large deflection of component plates are minimized and the assumption that only material nonlinearity takes place is valid, at least for a major part of the analysis.

Some of the material properties have been given in Sect. 2.3.4. The yield stress and the ultimate stress of the steel parts are 31.0 ksi and 44.0 ksi, respectively. The stress-strain relationship is almost trilinear, as obtained from a tension coupon testing. The strain hardening begins at 0.014 in./in. and the average strain hardening modulus is 500 ksi. For the reinforced concrete top flange
the stress-strain curve is that which has been discussed in Sect. 3.2.1,
Fig. 3.1b, with an ultimate compressive stress of $f'_c = 5.5$ ksi.

The discretization of this box girder into a mesh of finite
elements has been examined in Sect. 2.2.2 and 2.2.3 and the mesh
divisions adopted for the elastic analysis are given in Sect. 2.3.4.
The same mesh divisions are used here for the ultimate strength
analysis.

Continuing from the elastic analysis of Sect. 2.3.4, the
effective stresses $\sigma_e$ of all elements are computed for the theoretical
yield load at which the highest stressed element in the box girder
reaches yielding. An incremental load is then applied and the new
effective stresses under the total load are evaluated according to the
current values of the compliance matrix $[D^{ep}]$, the element stiffness
matrix $[k_e]$, and the global stiffness matrix $[k]$. For any of the
elements to each yielding, a scaling factor $\gamma$ can be determined by
using the formula (50)

$$r = \frac{\Gamma + \Gamma^2 + 4\sigma_{eI}^2 (F_y - \sigma_e^2)}{2 \sigma_{eI}^2}$$

$$\Gamma = \sigma_{eI}^2 - 2 \sigma_e (\Delta \sigma_e) - (\Delta \sigma_e)^2$$

(3.34)

where $\sigma_{eI}$ is the effective incremental stress due to the incremental
load and is computed by using Eq. 3.20. $F_y$ is the yield stress of
the material. $\Delta \sigma_e$ is the increment of the effective stress from $\sigma_e$,
due to the load increment. If at this stage a total of $m$ elements
are specified to reach yielding, then the scaling factor, $r_m$, for
the mth element is used to determine the magnitude of the load increment, \( P_2 = P_1 + r_m P_1 \).

This process is continued for all elements which are not yielded. For yielded elements as stresses approach the tensile strength of the material, the same procedure of scaling for each load increment can be applied. It is only necessary to replace the yield stress \( F_y \) in Eq. 3.34 by the tensile strength \( \sigma_{ult} \). The appropriate value of strain hardening modulus must then be used. For concrete elements, the same procedure can also be used with its yield curve as defined by Fig. 3.2b.

The load deflection curve of box girder D1 for the vertical deflection at the mid-span and under the web is given in Fig. 3.9. The test data are also shown. The correlation between the test data and the analytical results of this study is very good. Except for a few test points around the bend of the curve, which are higher than the actual static loads of the test (28), the computed curve practically coincides with the test data. No residual stress was considered in the analysis, and the actual magnitudes of these residual stresses are assumed to be small (28). The analysis indicates the effect of strain hardening at the ultimate load, just as the test results have shown. The unloading portion of the box girder behavior can not be described by the analysis. The incremental procedure is stopped at the ultimate load level when the incremental deflections are excessive corresponding to small incremental load.
Specimen D2 of Ref. 28 is also analyzed. This box girder has a thinner web and a more liberal spacing of transverse stiffeners when compared to specimen D1. The webs of the girder developed prominent tension fields and lateral deflections of the web plates were not small prior to the attainment of the ultimate load. The results of the analysis, shown as the load deflection curve in Fig. 3.10 deviate only slightly from the test results in the elastic range of behavior. The difference increases as the load approaches the highest value. At the ultimate load, the predicted value is 3% higher than the test data. This is regarded as quite acceptable considering that geometric nonlinearity is not incorporated in the analysis.

As it has been pointed out earlier, mesh divisions influence the results. Finer elements permit more accurate predictions. It has also been mentioned that the size of load increment affects the computed deflections and stresses. Smaller load increments give magnitudes closer to the measured ones. Box girder D1 is analyzed by using a relatively coarse mesh division but small load increments. The webs are each divided into three parts compared to five for Fig. 3.9. The top and bottom flanges between the webs are divided into two equal parts instead of four. Ten divisions along the half length of the box girder are used in place of 14. The ultimate strength obtained is the same as that shown in Fig. 3.8. This indicates that, for the determination of the ultimate strength of box girders, a relatively crude mesh division can be used so long as the load increments in the process of analyzing are reasonably small.
The inherent condition of the displacement method is that the predicted stresses may be less accurate than the deflections. To examine this, the variation of the normal stress at a point of box girder D1 is obtained and is plotted in Fig. 3.11 for comparison with measured values. The point examined is at the bottom flange, 3.75 in. away from mid-span. In Fig. 3.11, the data points are converted from the measured strains according to the stress-strain curve of the steel. Above 60 kips of applied load, the computed stresses increase because of strain hardening, and the values deviate from the measured stresses. In the range of elastic behavior and the first portion of the plastic flow, the computed stress agree very well with the test data. It is to be noted that the load increments in the inelastic analysis are quite small, as can be seen in this figure.

From the finite element analysis, the conditions of the box girder at all loads can be evaluated. Figure 3.12 shows an example which depicts the directions and magnitudes of the principal stresses in the elements of one web panel of girder D1. At the load magnitude of 56.84 kips, the element adjacent to the load has very high compressive stress and the lower four elements are under high tensile stress. All the elements next to the centerline of the girder have yielded. In fact, a number of elements have reached yielding, as is shown in Fig. 3.13. The spread of yielding can easily be traced throughout the entire loading sequence.

The significance of these illustrations on principal stresses and yield spreading is that, when the residual stresses are not
negligible and their magnitudes are known, their effects on the behavior of the box girder can be evaluated throughout the entire range of the box girder behavior. Coupled with the predicted deflections, these provide a reliable basis for design.
4. SUMMARY AND CONCLUSIONS

The ultimate strength and the load-deflection relationship of straight, rectangular box girders have been investigated. Instability of the box girder or its components is assumed to have been prevented. The loads on the girder are applied either through diaphragms or at junctions of webs and flanges. The finite element method is used with an automatic mesh generation program; and the displacement method is employed with the incremental load process.

In the investigation the elastic behavior of the box girders is first evaluated and compared with existing solutions so as to examine the procedure developed in this study. The inelastic analysis is carried out based on the results of this examination. A very brief example is given to show the capability of the procedure in developing aids for design and analysis.

Conclusions which can be drawn from this study are the following:

1. Two degrees of freedom per nodal point in the finite element formulation of box girder analysis can give accurate results compared to five degree of freedom formulation, if the out-of-plane plate bending is insignificant. For the same finite element discretization, the two degree of freedom system requires much less computer processing time.
2. For the box girders, fairly coarse mesh divisions provide fairly good results for deflection. For a better estimate of stresses, moderately fine mesh divisions are sufficient. Elements should have a low ratio of length to width, preferably not more than 2:1, depending on the fineness of the mesh division.

3. The finite element procedure incorporates the effects of shear lag, shear deformation, diaphragm rigidity and spacing, and boundary conditions. These can be used for the analysis of thin flange deck as well as box girders.

4. The procedure provides an efficient and accurate method for the evaluation of box girder deflections and stresses.

5. For composite box girders which have a concrete top flange, the selection of the box girder depth would affect the results of the analysis. Either the actual web depth or the centerline distance between the flanges may be considered as the box girder depth. Test results indicate a depth in between the two values.

6. The rigidity of diaphragm affects the behavior of box girders under torsion. The required thickness or rigidity of a diaphragm to ensure maintaining of cross-sectional shape can be easily estimated.

7. When torsional loads are applied between two diaphragms, the deflection and rotation of a box girder are relatively large only between these diaphragms. If torsional load is applied at a diaphragm, the deflection and rotation of the box girder are practically the same regardless of the diaphragm spacing.
8. Diaphragm spacing has very little effect on the behavior of box girders under flexural loading alone.

9. The load-deflection relationship of box girders can be obtained accurately by the procedure of this study if plate deflections of the web and flanges are not large. Girder deflections and corresponding stresses and strains can be determined for loads which only cause elastic responses as well as for loads which would cause large deflection of the girder and failure.

10. The ultimate strength or load carrying capacity of box girders can be predicted from the geometry and material properties of the box girder.

11. The estimated ultimate strength is higher than the actual load carrying capacity. The amount of overestimation depends on the finite element mesh division and the sizes of the incremental load. Finer mesh divisions and more important, smaller load increments result in better estimates.

The procedure developed in this study provides a powerful tool for the evaluation of stresses and deflection of straight, prismatic rectangular box girders. Residual stresses can be incorporated, so can the transverse and longitudinal stiffeners. The effects of the concrete deck thickness need to be studied further for inelastic analysis through dividing the deck thickness into layers of finite elements, a formidable task. Girders with trapezoidal cross-section, with taper, or with horizontal curvature should be examined. The incorporation of geometric nonlinearity also needs to be made to render the analysis more general. At the present, it appears that a
parametrix study on the influence of box girder geometry by using the procedure of this report would provide helpful information for the design, analysis, and erection of box girders for bridges, buildings, and other structures.
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Symmetric with respect to diagonal

α = \frac{a}{b}

β = \frac{b}{a}

Multiplier = \frac{t}{48}
### Table 2.2
Comparisons of Two Degree of Freedom and Five Degree of Freedom Results

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<td>Structure</td>
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<td>Box</td>
<td>Box</td>
<td>Box</td>
<td>Wide Flange</td>
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<td>Loading</td>
<td>Bending</td>
<td>Torsional</td>
<td>Bending</td>
<td>Torsional</td>
<td>Bending</td>
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<td>Division Along the Half Length</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>6</td>
<td>10</td>
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<td>Mesh</td>
<td>Coarse</td>
<td>Coarse</td>
<td>Medium</td>
<td>Medium</td>
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<td>2.2b</td>
<td>2.2b</td>
<td>2.2c</td>
<td>2.2c</td>
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<td>Maximum Deflection (in.)</td>
<td>2 DOF</td>
<td>0.04068</td>
<td>0.04556</td>
<td>0.04098</td>
<td>0.04323</td>
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<td>5 DOF</td>
<td>0.03976</td>
<td>0.04454</td>
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<td>0.04239</td>
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<td>Maximum Top Flange Stress ($\sigma_n$ ksi)</td>
<td>2 DOF</td>
<td>-0.2948</td>
<td>-0.3099</td>
<td>-0.2738</td>
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<td></td>
<td>5 DOF</td>
<td>-0.2821</td>
<td>-0.2912</td>
<td>-0.2793</td>
<td>-0.2896</td>
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<tr>
<td>Maximum Bottom Flange Stress ($\sigma_n$ ksi)</td>
<td>2 DOF</td>
<td>7.0549</td>
<td>6.5223</td>
<td>6.6931</td>
<td>6.6776</td>
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<td>CP Time for Solution (sec)</td>
<td>2 DOF</td>
<td>0.818</td>
<td>1.644</td>
<td>1.7139</td>
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<td></td>
<td>5 DOF</td>
<td>7.906</td>
<td>15.449</td>
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<td>Semi-band Width</td>
<td>2 DOF</td>
<td>24/164</td>
<td>24/164</td>
<td>38/168</td>
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<td>Total Degrees of Freedom</td>
<td>5 DOF</td>
<td>55/368</td>
<td>55/368</td>
<td>87/381</td>
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### TABLE 2.3

**EFFECT OF MESH DIVISION ALONG THE LENGTH**

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<td>FEM</td>
<td>16.35</td>
<td>88.20</td>
<td>344.1</td>
<td>4.667</td>
<td>7.000</td>
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<td>Beam Theory</td>
<td>18.19</td>
<td>132.59</td>
<td>1033.29</td>
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<td>12.000</td>
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<tr>
<td>%</td>
<td>89.9</td>
<td>66.5</td>
<td>33.3</td>
<td>77.7</td>
<td>58.3</td>
<td>29.2</td>
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<tr>
<td>FEM</td>
<td>4.37</td>
<td>21.30</td>
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<td>Beam Theory</td>
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<td>22.20</td>
<td>154.27</td>
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<td>%</td>
<td>99.5</td>
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<td>78.1</td>
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<td>FEM</td>
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<td>0.75</td>
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<td>%</td>
<td>102.0</td>
<td>90.0</td>
<td>66.9</td>
<td>81.7</td>
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* for finite element analysis \( \nu = 0 \)

# beam theory including shear deformations
TABLE 2.4 EFFECT OF DIVISIONS ACROSS THE FLANGES

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<td>Flange Normal Stress (b) (ksi)</td>
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</tr>
</tbody>
</table>

all stresses and displacements at \( x = \frac{L}{2} \)
### TABLE 2.5 EFFECT OF NUMBER OF DIVISIONS ACROSS THE WEB

<table>
<thead>
<tr>
<th>Number of Divisions Across The Web</th>
<th>Simple Bending</th>
<th>Pure Torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maximum Vertical Deflection (a) (in.)</td>
<td>Bottom Flange Normal Stress (b) (ksi)</td>
</tr>
<tr>
<td>1</td>
<td>0.03820</td>
<td>6.6847</td>
</tr>
<tr>
<td>2</td>
<td>0.03853</td>
<td>6.5152</td>
</tr>
<tr>
<td>3</td>
<td>0.03861</td>
<td>6.5439</td>
</tr>
<tr>
<td>4</td>
<td>0.03864</td>
<td>6.5487</td>
</tr>
<tr>
<td>5</td>
<td>0.03866</td>
<td>6.5518</td>
</tr>
<tr>
<td>6</td>
<td>0.03866</td>
<td>6.5536</td>
</tr>
<tr>
<td>Loading</td>
<td>Deflection (a) (in.)</td>
<td>2 DOF</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>----------------------</td>
<td>-------</td>
</tr>
<tr>
<td>Simple Bending</td>
<td>Deflection</td>
<td>1.429</td>
</tr>
<tr>
<td></td>
<td>Normal Stress (b) (ksi)</td>
<td>2.732</td>
</tr>
<tr>
<td>Pure Torsion</td>
<td>Deflection</td>
<td>2.009</td>
</tr>
<tr>
<td></td>
<td>Normal Stress</td>
<td>3.812</td>
</tr>
<tr>
<td>$n = \frac{E_s}{E_c} = 10$</td>
<td>Simple Bending</td>
<td>1.456</td>
</tr>
<tr>
<td>Same Box Girder with</td>
<td>Normal Stress</td>
<td>27.811</td>
</tr>
<tr>
<td>Transformed Steel Section</td>
<td>Pure Torsion</td>
<td>2.023</td>
</tr>
<tr>
<td></td>
<td>Normal Stress</td>
<td>38.586</td>
</tr>
<tr>
<td>D/t</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>t (in.)</td>
<td>0.240</td>
<td>0.102</td>
</tr>
<tr>
<td>Maximum Deflection (a)</td>
<td>0.02538</td>
<td>0.03251</td>
</tr>
<tr>
<td>Top Flange Normal Stress (b)</td>
<td>-0.2568</td>
<td>-0.2652</td>
</tr>
<tr>
<td>Bottom Flange Normal Stress (c)</td>
<td>4.1498</td>
<td>4.9733</td>
</tr>
<tr>
<td>Web Shear (d)</td>
<td>0.5830</td>
<td>1.2345</td>
</tr>
<tr>
<td>Shear Flow</td>
<td>0.1399</td>
<td>0.1481</td>
</tr>
<tr>
<td>Maximum Deflection (a)</td>
<td>0.002827</td>
<td>0.003839</td>
</tr>
<tr>
<td>Top Flange Shear (b)</td>
<td>0.02709</td>
<td>0.02524</td>
</tr>
<tr>
<td>Bottom Flange Shear (c)</td>
<td>-0.5058</td>
<td>-0.5281</td>
</tr>
<tr>
<td>Web Shear (d)</td>
<td>0.1901</td>
<td>0.4453</td>
</tr>
<tr>
<td>Shear Flow</td>
<td>0.04562</td>
<td>0.5344</td>
</tr>
</tbody>
</table>
Fig. 2.1 Finite Element Model and Element Coordinate Axes
Fig. 2.2 Box Girder Specimen and Mesh Division
Fig. 2.3 Wide Flange Beam

P = 10 kips

\[ L/2 = 50'' \]

W8x31

\[ E = 29,600 \text{ ksi} \]
Five Degrees of Freedom

Fig. 2.4 Comparison of Deflections from Two and Five Degree Freedom Formulation
Fig. 2.5 Normal Stress Distribution for Problem 2
(Cross-section at 50 in. from Left Support)
Fig. 2.6 Aspect Ratio and Mesh Division Along the Length
Fig. 2.7 Effect of Mesh Division Along the Length Under Simple Bending
Fig. 2.8 Effect of Mesh Division Along the Length Under Torsional Bending
Fig. 2.9 Relative Increase of Accuracy and Computational Time with Mesh Division
Fig. 2.10 Two Span Multicell Box Girder, Cross-section and Mesh Sizes
Fig. 2.11 Vertical Deflection Along the Longitudinal Line at the Top of the Loaded Web
Fig. 2.12 Vertical Deflection of Bottom Flange at Transverse Sections
Fig. 2.13 Longitudinal Distribution of Normal Stress ($\sigma_n$) Along a Line in Top Flange
Fig. 2.14 Stress and Deflection of W8 X 31
Fig. 2.15 Cross-section and Finite Element Models of a Thin Flange
Fig. 2.16 Top Flange Normal Stresses at a Transverse Section

\[ \bar{\sigma} = \frac{M_{\text{max}}}{h \cdot t \cdot B} \]

- Test Results
- Finite Elements
- Schmidt's Prediction

A = 78.5 cm
B = 26 cm
b = 25 cm
78.5 cm

DISTANCE FROM THE WEB

0.000 5.000 10.000 15.000 20.000 25.000
Fig. 2.17 Effect of Two Degree versus Five Degree Freedom Formulation on a Thin Flange
Fig. 2.18 Dimensions and Geometry of Girder D1
Fig. 2.19 Deflection Along the Length of Girder D1

(a) Finite Element Analysis

(b) Thin Walled Elastic Beam Theory

(c) Test Results
Finite Element Analysis

Thin Walled Elastic Beam Theory

Test Results

STRESS DISTRIBUTION AT X=86.25 DUE TO 18.0 KIP OF LOAD EC= 7.068

Fig. 2.20a Normal Stress Distribution due to a Torsional Loading on Girder D1 (86.25 in. from Left Support)

Scale

1 in. : 5 ksi

-95-
Finite Element Analysis

Thin Walled Elastic Beam Theory

Test Results

STRESS DISTRIBUTION AT X=56.25 IN. DUE TO 18. KIP OF LOAD \( EC=0.0 \)

Fig. 2.20b Normal Stress Distribution due to Simple Bending of Girder Dl (56.25 in. from Left Support)
Fig. 2.21 Effect of Diaphragm Thickness on Stresses and Deflection under Pure Torsional Loading

Deflection (inches)
(Stress (ksi))

Deflection under Web

Thickness of the Diaphragm

Web Shear at Midspan

$M_T = 5.0 \times 15 = 75.0$ k-ft.
Fig. 2.22 Principal Stresses in a Diaphragm
Fig. 2.23 Effect of Diaphragm Spacing on Deflection, Load at Mid-Span
Fig. 2.24 Effect of Diaphragm Spacing on Deflection, Load at Midspan, Exaggerated Scale.
Fig. 2.25 Effect of Diaphragm Spacing on Deflection, Load at Quarter Points
Fig. 2.26 Effect of Diaphragm Spacing on Deflection, Load at Quarter Points, Exaggerated Scale
Fig. 3.1 Stress-Strain Relationship for Steel and Concrete
Fig. 3.2 Yield Criteria for Steel and Concrete
Fig. 3.3 Graphical Representation of Incremental Tangent Stiffness Procedure
Fig. 3.4 Loading and Finite Element Discretization of a Steel I-Beam
Test Results from Ref. 67

Analytical Solution

Fig. 3.5 Load Deflection Curve for a Steel I-Beam
Fig. 3.6 Dimensions and Geometry of Specimens - M1 and M2
Fig. 3.7 Load versus Midspan Deflection - Specimen M1
First Signs of Noticeable Web Bulging in Panel 5

Fig. 3.8 Load versus Shear Stress at Centerline of Panel 5 - Specimen M2
Fig. 3.9 Load versus Midspan Deflection - Girder D1
Fig. 3.10 Load versus Midspan Deflection - Girder D2
Fig. 3.11 Load versus Normal Stress at 3.75 in. away from Midspan at the Bottom Flange
Fig. 3.12 Principal Stresses in the Web Adjacent to Loading Point

Scale: 0 10 20 ksi  Yielded Elements
Fig. 3.13 Yield Zones of the Web of Girder D1 under Different Load Magnitudes
REFERENCES


6. "Inquiry Into The Basis of Design and Method of Erection of Steel Box Girder Bridges", Report of the Committee, Appendix I, Department of Environment, Scottish Development Department, Welsh Office


10. Scordelis, A. C., "Analysis of Simply Supported Box Girder Bridges", Report No. SESM 66-17, Department of Civil Engineering, University of California, Berkeley, California, 1966


20. Scordelis, A. C., "Analysis of Continuous Box Girder Bridges", Report No. SESM 67-25, Department of Civil Engineering, University of California, Berkeley, 1967


44. Yilmaz, C., "Documentation for Program BOXGIR", Fritz Engineering Laboratory Report, Lehigh University, Bethlehem, Pa., (in preparation)


47. Lin, C. S., "Nonlinear Analysis of Reinforced Concrete Slabs and Shells", Report No. SESM 73.7, Department of Civil Engineering, University of California, Berkeley, 1973


NOMENCLATURE

a  half length of a rectangular element
b  half width of a rectangular element
[B] matrix relating nodal point displacements to element strains
B  semi-band width of the total stiffness matrix
[D] compliance matrix
[D^e] elasticity matrix
[D^{ep}] elastic-plastic compliance matrix
E  Young's modulus of elasticity
E_c Young's modulus of elasticity for concrete
Est strain hardening modulus
F  loading function
f'_c compressive strength of concrete
f'_t tensile strength of concrete
H  functional representing effective stress versus effective strain
H' slope of the effective stress-strain curve
K  yield condition
[K] total global stiffness matrix
[k_e] element stiffness matrix
[k_e]^g element stiffness matrix in global coordinates
[k_i] in-plane element stiffness matrix
[k_b] plate bending element stiffness matrix
L  span length of a girder
m,n  constants in Ramberg Osgood equation

N  total number of degree of freedom

[N]  shape function

N_i, N_j  individual terms in the shape function

[P]  load vector for the complete system

[ΔP]  load increment vector

scaling factor

r_i  scaling factor for the individual elements

r_m  scaling factor that will be used in the computations

[T]  transformation matrix

t  thickness of an element

{u}  element displacement vector

u, v  displacement of a point in x and y-direction, respectively

u_i, u_j  generalized nodal point displacements

w  displacement in z-direction

α_1, α_2  constants

γ_{xy}  shearing strain

[δ]  nodal point displacement for the complete system

\epsilon_x, \epsilon_y  strain in x-direction y-direction respectively

{ε}  strain vector

[ε_i]  incremental strain vector

[ε_o]  initial strain state

[ε_p]  equivalent plastic strain

\hat{ε}_p  equivalent plastic strain
\{e^P\} \quad \text{plastic strain vector}

\{\dot{e}^P\} \quad \text{plastic strain increment vector}

de^P_x, de^P_y \quad \text{plastic strain increments}

\sigma_x, \sigma_y, \sigma \quad \text{normal stresses}

\tau_{xy} \quad \text{shearing stress}

d\sigma_x, d\sigma_y, d\tau_{xy} \quad \text{increments of stresses}

\{\sigma\} \quad \text{stress vector}

\{\sigma\} \quad \text{increment of stress vector}

\{\sigma'\} \quad \text{deviatoric stress vector}

\{\dot{\sigma}'\} \quad \text{increment of deviatoric stress vector}

\{\sigma_o\} \quad \text{initial stress vector}

\sigma_e \quad \text{effective stress}

\sigma_o, \sigma_y \quad \text{yield stress level}

\sigma_1, \sigma_2 \quad \text{principal stresses in two perpendicular directions}

\sigma_{eI} \quad \text{increment in effective stress due to plastic stress increment}

\theta_x, \theta_y, \theta_z \quad \text{rotation about x, y, z axis respectively}

\nu \quad \text{Poisson's ratio}

d\lambda \quad \text{a constant in elastic-plastic compliance matrix}
VITA

The author was born in Uluborlu, Turkey on June 20, 1946, the second child of Mustafa and Hafize Yılmaz.

He was graduated from İzmir Atatürk High School in June 1963. He attended Middle East Technical University in Ankara, Turkey from September 1963 to June 1968, supported by a "Kennedy Scholarship", and received the B. S. degree in Civil Engineering.

He came to Lehigh University in September 1968 under a fellowship through the Institute of International Education and completed his M. S. Degree in Civil Engineering in October 1969. He worked as a graduate assistant at Middle East Technical University until coming back to Lehigh University in September 1971. He is supported by the Agency for International Development and involved with the project dealing with strength of composite box girders.

He is married to the former Fusun Özdamar and has one daughter, Ebru. The author will return to Turkey and join the staff of Middle East Technical University in Ankara.