An upper bound analysis of strip rolling by a velocity field of triangular regions.

Gordon, William A.
1984

Find more at https://preserve.lib.lehigh.edu/

This document is brought to you for free and open access by Lehigh Preserve. It has been accepted for inclusion by an authorized administrator of Lehigh Preserve. For more information, please contact preserve@lehigh.edu.
AN UPPER BOUND ANALYSIS OF STRIP ROLLING
BY A VELOCITY FIELD OF TRIANGULAR REGIONS

By

William A. Gordon

A Thesis
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Master of Science
in
Metallurgy and Materials Engineering

Lehigh University
1984
This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 11, 1984

(date)

Professor in Charge

Chairman of Department of Metallurgy & Materials Engineering
ACKNOWLEDGEMENTS

I would like to extend my sincere appreciation to all those who assisted me in completing this thesis. I am grateful to Bethlehem Steel Corporation for their support in this work as well as Dr. David Thomas and Dr. Bruce Somers for their financial assistance. I am also grateful to Peggy Kercsmar for the typing of this document.

I would like to thank my family, especially my mother and father, Mr. and Mrs. Walter F. Irmischer, for their support throughout my graduate school years.

Finally, I would like to extend my thanks to Dr. Betzalel Avitzur and the members of the Institute for Metal Forming. Without their combined efforts, much of this work would not have been possible.
# TABLE OF CONTENTS

NOMENCLATURE ........................................... vi
LIST OF FIGURES ........................................ ix
ABSTRACT ................................................ 1

1. INTRODUCTION AND BACKGROUND ..................... 2
   1.1 The Process ........................................ 2
   1.2 Background ......................................... 4
   1.3 The Upper Bound Technique ......................... 5

2. ASSUMED FLOW PATTERNS .............................. 7
   2.1 Continuous Region of Plastic Deformations ........ 7
   2.2 Triangular Velocity Field with Linear
         Rigid Body Motion .................................. 10
   2.3 Triangular Velocity Field with Rotational
         Rigid Body Motion .................................. 12

3. POWER TERMS .......................................... 16
   3.1 Continuous Region of Plastic Deformation .......... 16
   3.2 Triangular Velocity Field with Linear
         Rigid Body Motion .................................. 20
   3.3 Triangular Velocity Field with Rotational
         Rigid Body Motion .................................. 24

4. OPTIMIZATION PROCEDURE ............................ 28

5. RESULTS AND DISCUSSION ............................ 31

6. SUMMARY .............................................. 34

7. POSSIBLE FUTURE WORK ............................... 35
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>TABLES</td>
<td>36</td>
</tr>
<tr>
<td>FIGURES</td>
<td>38</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>50</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>52</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>57</td>
</tr>
<tr>
<td>APPENDIX C</td>
<td>62</td>
</tr>
<tr>
<td>VITA</td>
<td>68</td>
</tr>
</tbody>
</table>
NOMENCLATURE

\( \alpha \) angular position

\( \alpha_{ak} \) contact angle of element \( k \) with the roll (when linear motion of the triangle is assumed). Fig. 3

\( \alpha_2 \) full angle of contact between strip and roll

\( \alpha_k \) the angular position of the apex \( k \) of Fig. 4

\( \alpha_n \) neutral angle

\( \beta, \beta_{1k}, \beta_{2k} \) angles of the flanks of the triangle \( k \) of a rigid linear motion (see Fig. 4)

\( f\% \) percent forward slip

\( F_1 \) friction force between entrance and neutral point

\( F_2 \) friction force between neutral point and exit

\( j^*, j^* \) weighted total power consumption

\( j^*_k \) weighted power consumption for triangular element \( k \)

\( J^*, J^*_T \) total power consumption

\( J^*_k \) power consumption for triangular element \( k \)

\( k \) index of each specific triangle, \( k = 1, 2, 3 \ldots N \)

\( \lambda \) \( \lambda = \frac{t_k}{t_{k-1}} < 1 \)

\( L_k \) projection length of triangular element \( k \)

\( m \) constant friction factor

\( N \) number of triangles

\( r \) radial coordinate

\( r\% \) percent reduction in thickness
$R_0$ roll radius
$R_k$ the distance of the apex of triangle $k$ from the center of the roll
$S_k, S_{1k}, S_{2k}, S_{3k}$ length of the flanks of the triangle $k$ (see Fig. 4)
$\sigma_{xb}$ back tension
$\sigma_{xf}$ front tension
$\sigma_0$ flow stress of the material
$t$ thickness of strip
$t_0$ initial thickness of strip
$t_f$ final thickness of strip
$t_{k-1}$ incoming thickness of element $k$
$t_k$ outgoing thickness of element $k$
$\tau$ shear stress
$\theta$ angular position
$\theta_k$ the angular position of the apex of triangle $k$
$U$ roll velocity
$v$ velocity of strip
$v_0$ entering strip velocity
$v_f$ exiting strip velocity
$v_k$ linear velocity of the triangle at the exit from triangle $k$
$v_i$ velocity of prescribed surface
$\Delta v$ tangential velocity difference
$w$ width of strip
\( W_{\text{ext}} \) - power required to overcome external tractions

\( W_{\text{f}} \) - power required to overcome friction losses

\( W_{\text{in}} \) - internal power of deformation

\( W_{k} \) - total power of triangle \( k \)

\( W_{\Gamma_{1k}}, \ W_{\Gamma_{2k}} \) - power consumed along the surfaces of the triangle \( k \); surfaces \( \Gamma_{1k}, \Gamma_{2k} \) and the interface of the strip with the roll, \( \Gamma_{3k} \)

\( W_{s} \) - shear power losses

\( \omega_{k} \) - angular velocity of rotation of triangular element \( k \)

\( x_{k} \) - optimal apex position of triangular element \( k \)

A parameter defined by Eq. (2-15)
<table>
<thead>
<tr>
<th>No.</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Neutral Point and Friction Directions</td>
<td>38</td>
</tr>
<tr>
<td>2</td>
<td>Strip, Roll, and Assumed Deformation Pattern</td>
<td>39</td>
</tr>
<tr>
<td>3</td>
<td>Boundary of Velocity Discontinuity</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>Triangular Field with Linear Rigid Body Motion</td>
<td>41</td>
</tr>
<tr>
<td>5</td>
<td>Triangular Velocity Field with Rotational Rigid Body Motion</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>Total Weighted Power as a Function of the Number of Elements</td>
<td>43</td>
</tr>
<tr>
<td>7</td>
<td>Power Consumption as a Function of Relative Exit Velocity</td>
<td>44</td>
</tr>
<tr>
<td>8</td>
<td>Optimal Relative Exit Velocity as a Function of Friction</td>
<td>45</td>
</tr>
<tr>
<td>9</td>
<td>Weighted Power as a Function of Friction</td>
<td>46</td>
</tr>
<tr>
<td>10</td>
<td>Weighted Power vs. Relative Thickness</td>
<td>47</td>
</tr>
<tr>
<td>11</td>
<td>Split End of Steel Strip</td>
<td>48</td>
</tr>
<tr>
<td>12</td>
<td>Split End (Alligatoring) vs. Sound Flow</td>
<td>49</td>
</tr>
</tbody>
</table>
ABSTRACT

The process of strip rolling is analyzed using the upper bound technique. Two triangular velocity fields, one with triangles in linear rigid body motion and the other with triangles in rotational rigid body motion, are developed. The total power is determined as a function of the four independent process parameters (relative thickness, reduction, friction, and front-back tension). The results of these two velocity fields are compared with the established solution from Avitzur's velocity field of continuous deformation. Upon establishing the validity of the triangular velocity field as an approach to the strip rolling problem, recommendations are suggested on how this approach can be used to study the split end or alligatoring defect.
1. INTRODUCTION AND BACKGROUND

1.1 The Process

The mechanical shaping or forming of the final product represents a substantial investment in any metals industry. Speed, efficiency and accuracy are factors in the proper selection of a metalworking operation. In the manufacture of products of uniform cross-sectional area, the process of rolling is often chosen. It is a very versatile process as evident by the wide assortment of product shapes and sizes that can be produced. Yet, of all the rolling operations, strip rolling is performed more often than all the others combined.

The objective of strip rolling, where the width of the strip is much greater than its thickness, is to reduce the latter. The simple geometry of the process facilitates analysis and understanding. The concepts which evolve from a study of the strip rolling process can be used to enhance the efficiency of the rolling mill operation. Furthermore, this increased understanding can also indicate those processing conditions in which there is danger of forming defects. Avoiding those regions of danger, the formation of defects during strip rolling can be diminished.

In this work, an analysis of strip rolling is made for a Mises' material using the upper-bound technique. In the forthcoming study, the rolls are of identical radius \( R_0 \), surface conditions, applied power, and circumferential velocity \( U \).

The incoming strip is drawn through the rolls by the force of friction between the surface of the rolls and the strip (see Fig. 1).
As the strip is dragged into the gap between the rolls, it reduces in thickness from \( t_0 \) at the entrance to \( t_f \) at the exit. A front tension \((n_{x_f})\) and back tension \((n_{x_b})\) are applied to the process. The strip enters the region of deformation with velocity \( v_0 \). As the thickness reduces, the velocity of the strip increases to a final velocity, \( v_f \). Under normal rolling conditions, the strip moves slower than the rolls at the entrance \((v_0 < U)\) and faster than the rolls at the exit \((v_f > U)\). Therefore, a neutral point must exist where the velocity of the strip is equal to the velocity of the rolls \((v = U)\). The friction forces \( F_1 \) from the entrance to the neutral point aids the rolling process while the friction force \( F_2 \) opposes the rolling action. The required power for rolling is determined from the difference between the friction at the entrance side and the friction at the exit side \((F_1 - F_2)\).

The position of the neutral point depends upon the chosen rolling parameters for the process. For example, the larger the reduction attempted, the larger the friction \((\tau F_1)\) required for the rolling to proceed. Under those conditions, the neutral point will move towards the exit. When the neutral point reaches the exit \((v_f = U)\), the maximum possible reduction is achieved and the process becomes unstable. Larger reductions will produce skidding over the rolls. For the analysis and further discussion of the maximum possible reduction, the position of the neutral point, forward slip, and instability, see Chapter 13 of Ref (1).
1.2 **Background**

Von Karman was the first to try to mathematically model the rolling process\(^2\). Using the slug or free-body equilibrium approach, he derived a differential equation from which the roll force could be calculated. Orowan further characterized the work by applying the theories of plasticity as boundaries to the rolling equation\(^3\). The resulting differential equation has since led to many solutions, dependent upon the boundary conditions and assumptions applied. Different choices of friction conditions, material properties, etc., has led to many solutions for the roll pressure distribution (see partial list in Refs. 3 and 4).

Another approach which can be used to theoretically study metal forming operations is the upper-bound technique. One of the major advantages of this method is its ability to understand how processing conditions influence the formation of defects during metalworking processes. For example, the central burst defect during wire drawing and extrusion has been analyzed by Avitzur, et. al.\(^5\)\(^-\)8. Through the use of this approach, a better understanding has been developed of how the drawing conditions (i.e. reduction, die cone angle, friction, and the properties of the material) may cause this defect. From this knowledge, a criterion has been established that indicates under which processing conditions a central burst in drawn or extruded materials might occur. Using this criterion to avoid the danger region, the central burst defect in the manufacture of drawn or extruded parts
can be eliminated. Gordon and Van Tyne have used a similar approach to study the formation of central burst in closed die forging. A criterion has been established which suggests those forging conditions to avoid central burst formation during closed die forging.

There have been several analysis of strip rolling using the upper-bound method. Avitzur assumed a velocity field which accounts for the position of the neutral point. From this field he was able to provide equations for the maximum reduction during cold strip rolling as a function of the rolling parameters. Piispanen, Eriksson, and Piispanen propose a velocity field which assumes the region of deformation to be a curvilinear triangle which rotates as if it were part of the roll. Kuhn has used this approach of moving rigid body triangles in an upper-bound solution to study the defect of alligatoring or split ends during slab rolling.

In this work, two velocity fields of rigid body triangular regions are compared. From the results, recommendations are made on how a criterion may be established to prevent the alligatoring or split end defect during strip rolling.

1.3 The Upper Bound Technique

The upper-bound approach has been quite successful and is a well-documented method for analyzing metalworking processes. The methodology begins with an assumed material flow pattern, termed a kinematically admissible velocity field. For a velocity field to be kinematically admissible, it must satisfy the geometrical boundary conditions and the volume constancy requirement. The material may be divided up into different zones; the velocity of each zone is determined individually. The chosen flow pattern should be as realistic as possible.
Often it is necessary to include extra parameters into the analysis to approach this realistic velocity field. These extra parameters are termed pseudo-independent variables\(^\text{15}\) and are included in the derivation of the velocity field. As the power for the process is calculated, these are treated as independent. But the actual values which these variables possess are determined from those combinations which produce the lowest power (Principle of Minimum Energy).

Then, in turn, the strain rates \(\dot{\varepsilon}_{ij}\), internal powers of deformation \(\dot{W}_i\), shear power losses along surfaces of discontinuities \(\dot{W}_r\), friction power losses \(\dot{W}_f\), and the power to overcome external pressure tractions \(\dot{W}_{\text{ext}}\) are determined. All these power terms are calculated based on standard definitions used in the upper-bound method\(^\text{15}\). The sum of all these power terms, which are functions of the geometry and physical properties of the process, is set equal to the externally supplied power \(J^*\) which is needed to forge or work the material. According to the upper-bound theorem, the true externally supplied power will be less than or equal to the sum of these individual power terms which have been calculated based upon a kinematically admissible velocity field.

In effect, for any combination of geometrical and physical conditions (independent variables), the process itself determines the values of its pseudo-independent variables, and thus the best velocity field.

An excellent summary of the upper-bound approach as an application of limit analysis to metal forming processes is presented by Avitzur\(^\text{15}\).
2. ASSUMED FLOW PATTERNS

For the strip rolling process, three patterns are presented to model the material flow within the region of deformation between the rolls. The first will be a recapitulation of the original work done by Avitzur using a continuous field of deformation. In this analysis, the general pattern of flow in the deformation region is a converging accelerated flow from entrance to exit, with a surface of velocity discontinuity at the entrance to the deformation region.

In the other two velocity fields, the deformation zone is divided up into a series of individual elements, shaped as triangles, possessing rigid body motion. In the first velocity field these triangles are assumed to move in a linear fashion. For the other velocity field it is assumed that they rotate. The three velocity fields are termed:

I. Continuous region of plastic deformation with eccentric cylinders.
II. Triangular velocity field with linear rigid body motion.
III. Triangular velocity field with rotational rigid body motion.

A detailed description of each velocity field follows.

2.1 Continuous Region of Plastic Deformations

A continuous region of plastic deformations, separating the entrance from the exit, is described in Refs. 10-12 and in Fig. 2. The surface is a cylindrical surface of velocity discontinuity. The
first point of contact between the strip and the roll is defined by the angle

\[ \alpha_2 = \sqrt{\frac{t_f}{R_0}} \sqrt{\frac{t_0}{t_f} - 1} \]  

(assuming \( \alpha_2 \) is small). At this point of contact a tangential line to the roll surface is prescribed. Where the tangential line intersects with the plane of symmetry of the rolls and the strip, the apex or center of surface \( r \) is assumed. The region of plastic deformation is bounded by the surface \( r \) and the plane connecting the axes of symmetry of the rolls.

Upon entering the region of plastic deformation (crossing the \( r \) surface), the velocity of a point in the workpiece changes direction from parallel to the plane of symmetry to a converging flow towards the apex \( 0 \). Volume constancy dictates that along \( r \), the velocity towards the apex is (see Fig. 3)

\[ v = v_0 \cos \theta \]  

(2-2)

introducing a shear component parallel to the surface \( r \) of

\[ \Delta v = v_0 \sin \theta \]  

(2-3)

The angle \( \theta \) changes gradually from \( \theta = 0 \) at the plane of symmetry to \( \theta = \alpha_2 \) on the surface of the strip.

As each point within the deformation region moves towards the exit, its speed increases and its direction of flow becomes closer to
the longitudinal direction. Upon reaching the exit all points in the workpiece have reached the speed $v_f$ and flow uniformly in the longitudinal direction. Thus there is no assumed velocity discontinuity at the exit.

The thickness $(t)$ of the strip changes as a function of position $(\alpha)$ as

$$t = t_f + 2 R_0 (1 - \cos \alpha) \quad (2-4)$$

For small angles $(\alpha \ll 1)$, it can be stated that

$$\sin \alpha = \alpha \quad \text{and} \quad \cos \alpha = 1 - \frac{\alpha^2}{2} \quad (2-5)$$

substituting the approximation from Eq. (2-5) into Eq. (2-4) leads to

$$t = t_f + R_0 \alpha^2 = R_0 \left( \frac{t_f}{R_0} + \frac{\alpha^2}{t_f} \right) = t_f \left( 1 + \frac{R_0^2}{t_f} \right) \quad (2-6)$$

From volume constancy,

$$v t = v_f t_f = v_n t_n = Ut_n \quad (2-7)$$
where \( t_n \) is the thickness of the strip at the neutral point. Setting \( \alpha_n \) as the angular position of the neutral angle along the roll, the velocity of the strip in the deformation region becomes

\[
v = \frac{t_f}{R_0} \frac{R_0 + \alpha_n^2}{u} \frac{t_f}{R_0 + \alpha^2}
\]

(2-7)

The power terms associated with this velocity field are calculated in Section 3-1.

2.2 Triangular Velocity Field with Linear Rigid Body Motion

A triangular velocity field based on linear rigid body motion is described in Fig. 4. Due to the symmetry of the process, only the upper half of the strip deformation is described. In the assumed flow pattern, the deformation region is divided into a series of \( N \) elements which move with a rigid body motion. The deformation power is determined by the internal shearing that occurs between these elements. The element \( k \) is bounded by three surfaces, creating a triangle with base on the roll and apex on the plane of symmetry. The surface of contact between the roll and each element is approximated by a straight line. Toward the entrance side of the triangular element, all points are assumed to flow in a direction parallel to the plane of symmetry of the strip, with a constant linear velocity. From the principles of volume constancy

\[
v_{k-1} = \frac{v_f t_f}{t_{k-1}}
\]

(2-9)
Upon crossing the surface which divides the entrance region from the kth triangle, the direction and the magnitude of the velocity change. All points within the kth element are assumed to move as a rigid body in a direction parallel to the assumed linear surface of the roll.

Work done by Avitzur and Pachla prove, through the principles of volume constancy, that the surface of velocity discontinuity between two intersecting linear velocity fields is a straight line. Thus the surface \( \Sigma_{2k} \) is a straight line. If it is described with respect to the symmetry by the angle \( \beta_{2k} \), the magnitude of the velocity in the kth element can be determined from the volume constancy principle. It is required that the normal components of the velocities be equal in each zone along their common surface of velocity discontinuity. So \( \Sigma_{2k} \)

\[
v \sin \left( \beta_{2k} - \alpha_k \right) = v_{k-1} \sin \beta_{2k}
\]

(2-10)

where \( v \) is the velocity in the kth element and \( \alpha_k \) is defined as the average angle of contact which the kth element makes with the roll surface.

Substituting Eq. (2-1) and simplifying yields

\[
v = \frac{v_{k-1} \sin \beta_{2k}}{t_{k-1} \sin(\beta_{2k} - \alpha_k)}
\]

(2-11)

As the material crosses the surface, the velocity again changes direction and magnitude. After crossing this surface the material again flows parallel to the plane of symmetry with a velocity.
v_k. Its magnitude, as determined through volume constancy is

\[ v_k = \frac{v_f t_f}{t_k} \]  

(2-12)

The surface \( \Gamma \) is a straight line with an angle \( \beta \) with respect to the plane of symmetry. The magnitude of \( v \) within the element can again be found. Setting the normal components along the \( \Gamma \) surface equal, the velocity within the element is determined to be:

\[ v = \frac{v_f t_f}{t_k} \frac{\sin \beta_k}{\sin(\beta_k + \alpha_k)} \]  

(2-13)

Note that Eqs. (2-11) and (2-13) are found to be equal.

2.3 Triangular Velocity Field with Rotational Rigid Body Motion

A triangular velocity field based on rotational rigid body motion is illustrated in Fig. 5. This flow pattern is similar to the previously discussed linear rigid body motion.

A cylindrical coordinate system \((R, \phi)\) is assumed with its center located at the center of the roll. The surface of contact between the element and the roll is thus described as part of a cylinder. The deformation region is divided into a series of \( N \) elements. The material within the \( k \)th element is assumed to rotate as a rigid body about the center of the coordinate system at a constant angular velocity of \( \omega_k \).
As in the linear velocity field, the material approaching the \( \Gamma_{2k} \) surface of velocity discontinuity flows with a linear rigid body motion parallel to the plane of symmetry of the strip. The linear velocity, \( v_{k-1} \), is given by Eq. (2-9). Upon crossing the \( \Gamma_{2k} \) surface, the velocity changes to a rigid body rotational motion. When the material exits the element across the \( \Gamma_{1k} \) surface, a linear rigid body motion return with magnitude \( v_k \) (Eq. 2-12).

Further work on the characteristics of surfaces of velocity discontinuities by Avitzur and Pachla\textsuperscript{16} proves that the surface derived through volume constancy from the intersection of a linear and rotational rigid body motion is an arc of a circle. For the specific case where the rotational rigid body revolves about the center of the coordinate system and the direction of the linear motion is perpendicular to the \( \phi = 0 \) axis, Eq. (4.2.1.2) from Ref. [16] can be simplified to the form of

\[
(R \cos \phi + \frac{v_1}{\omega_k})^2 + R^2 \sin^2 \phi = R_{r}\r^2
\]  

(2-14)

where \( R_r \) is the radius of the circle with its center located at \( \left(-\frac{v_1}{\omega_k}, 0\right) \) in \((R, \phi)\) coordinates and \( v_1 \) is the magnitude of the linear velocity. The position of this surface of velocity discontinuity is fixed by locating its two endpoints. As can be seen in Fig. 5, one
endpoint is located along the surface of the roll; the other endpoint is along the plane of symmetry of the strip. Using these two points, the position of each surface is fixed (see Appendix A for derivation).

For the \( \Gamma_{1k} \) surface, this leads to

\[
\frac{R_{1k}}{R_0} = \sqrt{1 + 2 \frac{W_k}{R_0} \cos \alpha_k + \left( \frac{W_k}{R_0} \right)^2}
\]

where

\[
W_k = \frac{\left( \frac{R_k}{R_0} \right)^2}{\frac{t_k}{R_0}}
\]

From the geometry

\[
\cos \alpha_k = 1 - \frac{1}{2} \left( \frac{t_k}{R_0} \right) + \frac{1}{2} \left( \frac{t_f}{R_0} \right)
\]

\[
\frac{R_k}{R_0} = \sqrt{\left(1 + \frac{1}{2} \frac{t_f}{R_0}\right)^2 + \left( \frac{x_k}{R_0} + \sin \alpha_k \right)^2}
\]
Similarly, for the $\Gamma_{2k}$ surface

$$
\frac{R_{\Gamma_{2k}}}{R_0} = \sqrt{1 + 2 \frac{W_{k-1}}{R_0} \cos \gamma_{k-1} + \left(\frac{W_{k-1}}{R_0}\right)^2}
$$

where

$$
\frac{W_{k-1}}{R_0} = 1 - \left(\frac{R_k}{R_0}\right)^2
$$

and also from the geometry

$$
\cos \gamma_{k-1} = 1 - \frac{1}{2} \frac{t_{k-1}}{R_0} + \frac{1}{2} \frac{t_f}{R_0}
$$

(2-19)

Knowing the position of the two surfaces of velocity discontinuities, the shear power losses may be calculated (see section 3.3).
3. POWER TERMS

3.1 Continuous Region of Plastic Deformation

As a result of the assumed velocity field, there are four power terms which make up the total energy required to perform the strip rolling process. They are:

1) Shear power losses along the surface of velocity discontinuity
2) Internal power of deformation
3) Friction power losses
4) Power associated with the front and back tensions

The complete derivation of the power terms is provided in Ref. [17] and will not be discussed in this paper.

The shear losses along the surface of velocity discontinuity (see Fig. 2) are expressed as

\[ W_{s1} = \frac{2}{\sqrt{3}} \sigma_0 \dot{U} t_f (1 + \frac{R_0}{t_f} n^2) \sqrt{\frac{t_f}{R_0}} \sqrt{\frac{t_0}{t_f}} - 1 \]  

Accounting for the ideal portion of the power and neglecting the shear losses in the region of plastic deformation, the internal power of deformation becomes

\[ W_{i1} = \sqrt{3} \sigma_0 \dot{U} t_f (1 + \frac{R_0}{t_f} n^2) \frac{t_0}{t_f} \]  

16
Friction losses, calculated to take into account the position of the neutral point, are determined to be

\[
W_{s2} = \frac{2}{3} m \sigma_0 \dot{U} R_0 \left\{ (1 + \frac{R_0}{t_f} \alpha_n^2) \sqrt{\frac{t_f}{R_0}} 
+ (2 \tan^{-1} \sqrt{\frac{R_0}{t_f}} \alpha_n - \tan^{-1} \sqrt{\frac{t_0}{t_f}} - 1) \right\}
\]

where \( m \) is defined as the constant friction factor.

Lastly, the power associated with the front and back tensions are determined to be

\[
W_{ab} = \dot{U} t_f (1 + \frac{R_0}{t_f} \alpha_n^2) (\sigma_{xb} - \sigma_{xf})
\]

The externally supplied power, \( J^* \), is set equal to the sum of all these individual power terms. Making the simplification that

\[
\tan^{-1} \sqrt{\frac{R_0}{t_f}} \alpha_n = \sqrt{\frac{R_0}{t_f}} \alpha_n
\]
\[ J^* = \frac{2}{ \sqrt[3]{3} } \sigma_o \cdot t_f \left\{ \left[ 1 + \left( \frac{R_0}{R} \right)^{\alpha_n} \right]^{\frac{1}{2}} \left[ \left( \frac{t_0}{t_f} \right)^{\alpha_n} + \left( \frac{t_0}{t_f} \right)^3 \right] \right\} 
\]

\[ + \frac{\sigma_x b - \sigma_x f}{(2/\sqrt{3}) \sigma_o} - m \sqrt{\frac{R_0}{t_f}} \tan^{-1} \sqrt{\frac{t_0}{t_f} - 1} \]

\[ + m \sqrt{\frac{R_0}{t_f}} \left[ 2 \left( \frac{R_0}{t_f} \right)^{\alpha_n} + \sqrt{\frac{t_0}{t_f} - 1} \right] \]  

This is the same as Eq. (15.18b) found in Ref. [17].

In functional form \( J^* \) can be expressed as

\[ \frac{J^*}{\sqrt[3]{3} \sigma_o \cdot t_f} = \left( \frac{t_0}{t_f}, \frac{t_f}{R_0}, m, \frac{\sigma_x b - \sigma_x f}{(2/\sqrt{3}) \sigma_o}, \text{ and } \alpha_n \right) \]  

The required energy for rolling is determined as a function of reduction \( t_0/t_f \), roll radius \( t_f/R_0 \), friction conditions \( m \), forward and backward tension \( (\sigma_x b - \sigma_x f)/(2/\sqrt{3}) \sigma_o \) and the parameter \( \alpha_n \), which controls the position of the neutral point. Under a set of constant process conditions (independent variables), \( \alpha_n \) is initially treated as an independent variable. But the value which \( \alpha_n \) comes to actually possess and thus the exact position of the neutral point will be, by the Principle of Minimum Energy, that value which requires the minimum amount of externally supplied power for the process to occur.
In effect, the process itself determines the value of \( \alpha_n \) and thus the best velocity field. Parameters like \( \alpha_n \) which have these characteristics are termed pseudo-independent parameters as distinct from the independent parameters over which the operator has direct control.

So, to minimize \( J^* \), Eq. (3-6) is differentiated with respect to \( \alpha_n \) and the result is set equal to zero. The actual position of the neutral point which minimizes the total power becomes

\[
\alpha_n = \frac{1}{2} \sqrt{\frac{t_f}{R_0}} \left\{ \tan^{-1} \left( \sqrt{\frac{t_o}{t_f}} - 1 \right) - \frac{1}{4} \frac{t_f}{R_0} \left[ i \frac{t_o}{t_f} \right] \right\}
\]

(3-8)

similar to Eq. (15.19) in Ref. [17].

Placing the value of \( \alpha_n \) back into the expression for \( J^* \), the externally-supplied power becomes a function of the rolling parameters

\[
J^* = \frac{2}{\sqrt{3}} \frac{t_o}{t_f} \left\{ \frac{1}{2} \left[ \tan^{-1} \frac{t_o}{t_f} + \frac{1}{4} \right] \right\} \left[ \frac{t_o}{R_0} - 1 + \frac{\frac{x_b - x_f}{2}}{\frac{\sigma_o}{3}} \right]
\]

(3-9)

Same as Eq. (15.18c) in Ref. [17].
3.2 Triangular Velocity Field with Linear Rigid Body Motion

The total power consumed around a single element is found by the summation of three power terms; the two power terms accounting for the shear losses along the $\Gamma_{2k}$ and $\Gamma_{1k}$ surfaces respectively and the friction losses between the element and the surface of the roll.

Work by Avitzur and Pachla shows an expression for the shear power losses along the surface of velocity discontinuity between two linear velocity fields. For the specific case where the direction of one linear motion is parallel to the x-axis, the power losses as defined by Eq. (3.4.3) of Ref. [16] simplify to

$$W_s = \frac{\alpha}{\sqrt{3}} \left[ \frac{v_2^2}{v_1} - 2 \frac{v_2}{v_1} \cos \alpha \right] (x_B - x_A)$$

(3-10)

where $v_1$ is the magnitude of the velocity which is parallel to the x-axis. Likewise, $v_2$ is the magnitude of the velocity in the other zone. $\alpha$ is the angle between the two velocities and $x_A$, $x_B$ are the x-coordinates of the surface of velocity discontinuity.

Applying Eq. (3-10) to the $\Gamma_{2k}$ surface (see Appendix B for derivation), the resulting shear losses become

$$W_{s_{2k}} = \frac{2^3}{2^3} \mu f \frac{1}{2} \frac{v}{U} \frac{R_0}{t_{k-1}} \left[ \frac{L_k}{R_0} - \frac{x_k}{R_0} \right]^2 + \left( \frac{t_{k-1}}{R_0} \right)^2 \frac{\alpha_k}{R_0}\cot \alpha_k \right]$$

(3-11)
Similarly, along the $\Gamma_{1k}$ surface

$$W_{\text{fr}} = 2 \frac{\tau_0}{\sqrt{3}} \frac{V_f}{U} \frac{R_o}{t_k} \left[ \frac{x_k}{R_o} + \frac{2^{t_k}}{2R_o} \cot \alpha_k \right]$$  (3-12)

The friction losses are calculated next. They follow the general expression

$$W_{\text{fr}} = 2 \frac{\tau_0}{\sqrt{3}} m \int \Delta v dS = \frac{2}{3} \frac{\tau_0}{m} U S_{3k}$$  (3-13)

From the geometry

$$S_{3k} = \frac{(t_k - t_{k-1})}{2 \sin \alpha_{ak}}$$  (3-14)

The tangential velocity difference along the $\Gamma_{3k}$ surface is equal to the velocity of the roll ($U$) minus the velocity of the element (Eq. (2-13)). This can be expressed in the form of

$$\Delta v = U \left[ 1 - \frac{V_f}{U} - \frac{t_f}{t_k} \right] \frac{R_o}{R_o} \left[ 1 - \frac{x_k}{t_k} \frac{\cos \alpha_{ak}}{2 \sin \alpha_{ak}} \right]$$  (3-15)

Substituting Eqs. (3-14) and (3-15) into Eq. (3-13), the friction losses are

$$W_{\text{fr}} = 2 \frac{\tau_0}{\sqrt{3}} \frac{V_f}{U} \frac{R_o}{t_k} \left[ \frac{R_o}{R_o} - \frac{x_k}{t_k} \frac{\cos \alpha_{ak}}{2 \sin \alpha_{ak}} \right]$$  (3-16)
So, the total power consumed for the kth element, $J^*_k$ is

$$J^*_k = \dot{W}_{s12k} + \dot{W}_{s11k} + \dot{W}_{s3k}$$  \hspace{1cm} (3-17)

where the individual terms are given by Eqs. (3-11), (3-12) and (3-16) respectively.

In functional form, the relative power, $j_k^*$, can be stated as

$$j_k^* = \frac{J^*_k}{\frac{2}{\sqrt{3}} U t_f} = f\left(\frac{t_0}{t_f}, m, \frac{t_f}{R_o}, \frac{v_f}{U}, \frac{t_k}{R_o}, \frac{t_{k-1}}{R_o}, \frac{x_k}{R_o}\right)$$  \hspace{1cm} (3-18)

The total relative power $j^*_T$ for the rolling is equal to the sum of the individual elements

$$j^*_T = \sum_{k=1}^{N} j_k^*$$  \hspace{1cm} (3-19)

In the case of the total power for one element, there is only one pseudo-independent parameter, the relative position of the element's apex, $x_k/R_o$, along the axis of symmetry of the strip. For a fixed set of independent parameters ($t_f/R_o$, $m$, $t_0/t_f$), fixed values of the elements intersection with the roll ($t_k/R_o$, $t_{k-1}/R_o$), and a particular value of the forward slip ($v_f/U$), the optimal position of $x_k/R_o$ may be found. (Note that the optimization procedure of $v_f/U$, $t_{k-1}/R_o$ and $t_k/R_o$ for the triangular velocity fields is explained in section 4.1).
The derivation of \( J_k^* \) is taken with respect to \( x_k \). (See Appendix C for the derivation). Setting this expression equal to zero, the optimal value of \( x_k/R_o \) in the triangular velocity field with linear rigid body motion is

\[
\left( \frac{x_k}{R_o} \right)_{opt} = \frac{t_k}{2R_o} \left\{ -\cot \alpha_{ak} + \frac{1}{\lambda \sin^2 \alpha_{ak}} \left( \frac{1 + \frac{1 - \lambda}{\lambda} \cdot m}{1 + \frac{1 - \lambda}{\lambda} \cdot m} \right) \right\} \tag{3-20}
\]

where \( \lambda = \frac{t_k}{t_k-1} \) and the plus sign under the square root applies when

\[
\frac{v_f}{t_f} \frac{t_f}{t_k} \cos \alpha_{ak} + 2 \frac{x_k}{t_k} \sin \alpha_{ak} > 1 \tag{3-21}
\]

If however, when the plus sign is applied and the inequality is not satisfied, then the minus sign should be applied and the term on the left at the inequality must become smaller (not greater) than one. This restriction occurs as a result of the change in the frictional force direction when the neutral point is passed.

The resulting power for one element becomes

\[
J_k^* = \frac{2}{\sqrt{3}} \dot{\gamma}_o t_f \left\{ \frac{v_f}{U} \left[ \frac{(1+\lambda)^2 + (1-\lambda)^2 m}{\lambda \sin^2 \alpha_{ak}} - 2 \cot \alpha_{ak} \right] \right\} + \frac{1}{2} \frac{t_k-1}{R_o} \frac{\lambda - 1}{\sin^2 \alpha_{ak}} m \tag{3-22}
\]

where the inequality of Eq. (3-21) applies to the selection of the sign for the last term.
3.3 Triangular Velocity Field with Rotational Rigid Body Motion

Similar to the linear rigid body motion discussed in the previous section, the total power consumed around a single element in a rotational rigid body motion, is the sum of three power terms. Those are the two power terms attributed to the shear losses along the \( \Gamma_{2k} \) and \( \Gamma_{1k} \) surfaces respectively and the friction losses between the element and the surface of the roll.

Further work by Avitzur and Pachla shows that for a rotational rigid body motion intersecting with a linear rigid body, the shear power losses along the surface of velocity discontinuity \( \Gamma_k \) is as follows: (Eq. (4.4.3) from Ref. [16])

\[
W_{s_{\Gamma_k}} = \frac{a_0}{\sqrt{3}} R_{\Gamma}^2 \omega_k \sin^{-1} \left[ \frac{X_A + \omega_k}{R_{\Gamma}} \right] - \sin^{-1} \left[ \frac{X_B + \omega_k}{R_{\Gamma}} \right]
\]

where \( X_A \) and \( X_B \) are the two points through which the shear losses are to be calculated. \( \omega_k \) is the angular velocity of the rotating element and \( v_1 \) is the magnitude of the linear velocity approaching the surface of velocity discontinuity, \( R_{\Gamma} \) is the radius of the surface \( \Gamma \). Applying this equation to the \( \Gamma_{2k} \) surface results in

\[
W_{s_{\Gamma_{2k}}} = \frac{a_0}{\sqrt{3}} \frac{v_f}{U} \left[ \frac{R_{\Gamma_{2k}}^2}{R_0^2} \right] \left[ 1 - \left( \frac{R_k^2}{R_0^2} \right) \right] \sin^{-1} \left[ \frac{R_{\Gamma_{2k}}^2}{R_0^2} \right]
\]
Similarly along the $\Gamma_{1k}$ surface

$$W_{si_{1k}} = \frac{2 \gamma}{\sqrt{3}} \frac{U}{U} \frac{\frac{t_f}{2}}{\frac{R_{1k}}{2}} \left\{ \sin^{-1} \left[ \frac{-1 + \frac{t_f}{2} \frac{R_{1k}}{R_o} - \frac{W_{k-1}}{R_o}}{\frac{R_{1k}}{R_o}} \right] - \sin^{-1} \left[ \frac{-1 + \frac{t_f}{2} \frac{R_{1k}}{R_o} - \frac{W_k}{R_o}}{\frac{R_{1k}}{R_o}} \right] \right\}$$

(3-25)

The friction losses between the rotating element and the roll for one element is

$$\dot{W}_{si_{3k}} = \int_{\theta=a_k}^{\alpha_{k-1}} m \frac{\sigma_0}{\sqrt{3}} | \Delta v | \ R_o \ d \theta$$

(3-26)

where the tangential velocity difference between the element and the roll is

$$| \Delta v | = | \omega_k \ R_o - \dot{U} |$$

(3-27)

Simplifying this expression and substituting Eq. (A-6) for $\omega_{kl}$, the friction losses become
Similar to the linear rigid body motion velocity field, the position of the apex, \( x_k/R_0 \), a pseudo-independent parameter, is a function of the power of the element only. Determination of this value is obtained by differentiating \( J^* \) with respect to the parameter, setting the derivative equal to zero then solving for the pseudo-independent parameter which is termed the optimal value. Unfortunately, there is difficulty in solving for \( (x_k/R_0)_{opt} \) explicitly. Therefore, numerical methods for optimization are used. The optimization procedure is discussed in the next section.

As in the linear velocity field, the total power consumed for the kth element, \( J_k^* \) is

\[
J_k^* = W_{s_f}^{2k} + W_{s_f}^{3k} + W_{s_f}^{3k}
\]

where the individual terms are given by Eqs. (3-24), (3-25), and (3-28).

In function form, the relative power \( j_k^* \) of the kth element can be stated as

\[
j_k^* = \frac{J_k^*}{J_k} = f\left(\frac{t_0}{t_f}, \frac{m}{R_0}, \frac{t_f}{U}, \frac{v_f}{R_0}, \frac{t_k}{R_0}, \frac{t_{k-1}}{R_0} \frac{x_k}{R_0}\right) \quad (3-30)
\]
The total relative power $j^*$ for the rolling is equal to the sum of the individual elements

$$ j^* = \sum_{k=1}^{N} j_k^* $$

(3-31)
4. OPTIMIZATION PROCEDURE

As stated in the previous section, the total power for the process as it is described by either of the two triangular velocity fields is equal to the sum of the powers of the individual triangles. This power \( J^* \) (for either triangular velocity field) can be presented symbolically as

\[
J^* = \frac{1}{2} \int \left( \sigma_0 U f \right) \text{d}t
\]

\[
= f \left( t_f, t_0, m, \frac{\sigma_b - \sigma_f}{\sigma_0}, & N, \right)
\]

\[
& t_1, t_2, t_3, \ldots, t_{k-1}, t_k, \ldots, t_{n-1},
\]

\[
& x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots,
\]

\[
& x_{n-1}, x_n, \frac{v_f}{U}, &
\]

The first four parameters are the independent process parameters, the same as existed in the continuous field equation, Eq. (3-9). The rest of the parameters are pseudo-independent variables, subject to optimization.

The optimization method used to determine these values can be described as four nested procedures. The inner procedure is the optimization of the power of one triangle with respect to its \( x_k \). As stated in sections 2.2 and 2.3, the position of \( x_k \) that provides the
minimum power consumption for the individual triangle \( k \) is dependent on the positions of the apexes \( t_{k-1}/R_0 \) and \( t_k/R_0 \) of the individual triangle only. This position is not dependent on the values of \( t_k/R_0 \) or \( x_k/R_0 \) for other triangles, nor is it a function of \( v_f/U \). So, only when \( t_{k-1}/R_0 \) or \( t_k/R_0 \) are changed for that particular triangle, a new optimal value of \( x_k/R_0 \) should be sought. For the triangular field of linear rigid body motion, \( x_k/R_0 \) is expressed explicitly. For rotational rigid body motion, \( x_k/R_0 \) is subject to optimization by numerical methods.

The next procedure outward is the optimization with respect to \( t_k/R_0 \). In this loop, the values of \( t_k/R_0 \)'s are searched for all the triangles so as to minimize the total \( j^* \). This requires that there be a particular set of \( t_k/R_0 \)'s, that when the total power for the process is calculated, it is found to be minimum. This requires a multidimensional gradient technique which ensures the satisfaction of all the imposed boundary conditions.

The third loop outward is optimization of \( v_f/U \), the relative exit velocity. Similar to the neutral angle of the continuous velocity field, the total power of the process is subject to minimization with respect to \( v_f/U \). Typical values of \( v_f/U \) lie in the range of 1.0 to 1.1. \( v_f/U \) is related to the neutral angle through the equation

\[
\frac{v_f}{U} = 1 + \frac{R_0}{t_f u_n}^2
\]  

(4-1)
Finally, in the outer most loop $N$, the number of triangles, is varied until the determined power begins to increase. The optimum number of triangles is then assumed to be found.
5. RESULTS AND DISCUSSION

Graphs are presented to illustrate the behavior of $j^*_T$ with respect to the pseudo-independent parameters (Figs. 6 & 7), and with respect to the independent parameters (Figs. 8, 9 & 10). A comparison is made between the results generated from the continuous velocity field and those from the two triangular velocity fields. Please note that the graphs presented for the triangular velocity field with rotational rigid body motion are based on preliminary studies of equations not in their final form. The equations presented in this text are in their final form and programming and execution runs are needed in order to generate new, more precise graphs for presentation.

In Fig. 6, for rigid body motion, the weighted power, $j^*_T$ is the ordinate, while the number of elements or triangles, $N$, is the abscissa. For the independent parameters indicated ($t_0/R_0 = 0.01$, $t_f/t_0 = 0.7$, $m = 0.2$ and $\frac{\sigma_{xb} - \sigma_{xf}}{2} = 0.0$), the triangular field of rotational motion yielded the lowest power for 10 elements. The triangular field of linear motion was minimized when 7 elements occur. This optimal relative power from linear rigid body motion is 0.51677, a little lower than the continuous solution of 0.53286. The preliminary results of the rotational field promise an even lower solution. Figure 6 illustrates that the triangular velocity fields do indeed provide a lower solution than the continuous velocity field.
Figure 7 shows the existence of the optimal relative exit velocity. For the same values of the independent process parameter as Fig. 6, there is a neutral angle where the velocity of the strip is equal to the velocity of the roll. With the values of $v_f/U$ being slightly larger than 1.0, indicating that there is a small degree of forward slip in rolling under these assumed conditions, the percent forward slip can be calculated from

$$f\% = \left( \frac{v_f}{U} - 1 \right) \times 100 \quad (5-1)$$

Table I summarizes the results of optimization of both the continuous and linear rigid body velocity fields. The values of $t_{k-1}/R_0$, $t_k/R_0$, and $x_k/R_0$, together with the individual weighted power consumed at each element (for 7 elements) in linear rigid body motion is also presented.

The validity of the triangular velocity field is further supported by studying the effects of the independent process parameters on the relative exit velocity and the optimal power, $j^*_k$. Figure 8 plots the relative exit velocity as a function of the friction factor, $m$ (with the previous values of $t_0/R_0$, $t_f/t_0$, and $\frac{a_{xb} - a_{xf}}{a_0} \frac{2}{\sqrt{3}}$ for both the continuous and triangular velocity fields. The general trend indicates that as the friction between the strip and the roll increases, the relative exit velocity increases indicating that the process becomes more stable. For low values of friction, $v_f/U$ approaches 1, the neutral angle approaches the exit, and rolling tends toward instability. Note that both the continuous and triangular velocity fields have the same trends.
The optimal number of elements for each triangular velocity field is given in brackets for the linear motion and parenthesis for the rotational motion.

Figure 9 displays the power $j^*$ as a function of friction under the same rolling conditions. The higher the friction, the larger the power needed to perform the process. Again, both the continuous and triangular fields illustrate the same trends, but the resulting power for the triangular fields are lower.

Figure 10 shows the weighted power as a function of the relative thickness $t_0/R_0$. In general, as the roll radius, $R_0$, decreases ($t_0/R_0$ increases), the length of contact between the roll and the strip decreases. As this length decreases, the power needed to perform the rolling also decreases. Again, this trend is shown in the results and again, the triangular field yields a lower solution.
6. SUMMARY

The upper bound technique has been used to analyze the process of strip rolling. Two triangular velocity fields, one with triangles in linear rigid body motion and the other with triangles in rotational rigid body motion, are developed. The total power, $J^*$, is determined as a function of the four independent process parameters (relative thickness, reduction, friction, and front-back tension) and the pseudo-independent parameters. After optimization, the results of these two velocity fields are compared with the established solution from the velocity field of continuous deformation. Though the optimization procedure of the triangular fields is indeed much more complex, these two fields provide a similar solution to the older velocity field. This establishes the validity of the triangular velocity field as an approach to the strip rolling problem.

The goal of the triangular velocity field in strip rolling is not simply to provide yet another analysis of strip rolling. The optimization procedure is far too complex to justify the use of the field in this manner. The strength of the approach is its ability to analyze the formation of the split end or alligatoring defect in the strip rolling process, as well as the ability to study other defects. For example, this study can serve as a basis for further work as described next.
7. POSSIBLE FUTURE WORK

Figure 11 shows the alligating of a steel strip after being rolled. The defect initiates as a crack, forming along the central plane of the deformed material (the plane connecting the axis of symmetry of the rolls). As the rolling process proceeds, the two "halves" of the material separate from each other and alligating occurs.

The general pattern of flow expected during strip rolling with split ends vs. sound flow is described in Fig. 12. In the top half, strip rolling with sound flow is described by use of a single rotational triangle. One triangle is also used in the bottom half for strip rolling with split ends. Note that for sound flow, the position of the apex $x_k$, along the axis of symmetry of the strip, is positive. If the surface of velocity discontinuity, $u_k$, is removed, and the position of the apex is allowed to be negative, the triangle $k$ would continue to rotate around the roll axis and split ends would develop. Should more than one triangle be used, this simplification in the velocity field will be adapted to the last triangle.

So, with the triangular velocity fields, a comparison can be made of how the rolling parameters influence the formation of the split end defect. In comparing both sound flow and flow with split ends, a criterion could also be developed for the range of combination of the independent parameter that allows rolling without split ends.

Thus, the analysis of split ends justifies the application of a triangular velocity field to the problems of strip rolling.
### TABLE I - RESULTS OF OPTIMIZATION OF CONTINUOUS AND TRIANGULAR LINEAR RIGID BODY VELOCITY FIELDS

**Rolling Conditions:**

\[
\frac{t_f}{t_0} = 0.7
\]

\[
m = 0.2
\]

\[
\frac{\sigma_{xb} - \sigma_{xf}}{2 \sigma_0} = 0.0
\]

<table>
<thead>
<tr>
<th></th>
<th>Avitzur's Continuous Field</th>
<th>Linear Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{v_f}{U} )</td>
<td>1.0451</td>
<td>1.0478</td>
</tr>
<tr>
<td>( \frac{j_f}{T} \times 10^3 )</td>
<td>0.53286</td>
<td>0.51595</td>
</tr>
<tr>
<td>( \alpha_n )</td>
<td>1.7767 \times 10^{-2}</td>
<td>X</td>
</tr>
</tbody>
</table>

36
<table>
<thead>
<tr>
<th>element ( k )</th>
<th>( \frac{t_k}{R_0} \times 10^{-3} )</th>
<th>( \frac{x_k}{R_0} \times 10^{-3} )</th>
<th>( j_k^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>9.0640</td>
<td>4.2031</td>
<td>0.16384</td>
</tr>
<tr>
<td>2</td>
<td>8.3641</td>
<td>3.8250</td>
<td>0.12213</td>
</tr>
<tr>
<td>3</td>
<td>7.8460</td>
<td>3.5334</td>
<td>0.08794</td>
</tr>
<tr>
<td>4</td>
<td>7.4731</td>
<td>3.3032</td>
<td>0.5967</td>
</tr>
<tr>
<td>5</td>
<td>7.1937</td>
<td>3.9631</td>
<td>0.04002</td>
</tr>
<tr>
<td>6</td>
<td>7.0466</td>
<td>3.8999</td>
<td>0.02763</td>
</tr>
<tr>
<td>7</td>
<td>7.0000</td>
<td>3.7548</td>
<td>0.01555</td>
</tr>
</tbody>
</table>

\[ j^*_T = 0.51595 \]
\[
F_{\text{FRONT PULL}} - F_{\text{BACK PULL}} = F_{\text{DRIVING FORCES}} - F_{\text{OPPOSING FORCES}}
\]

Net driving force: \( F_{\text{FRONT PULL}} - F_{\text{BACK PULL}} = F_{1} - F_{2} \)

\[v_0 < \dot{U} < v_f\]
\[\frac{v_f}{v_0} = \frac{t_0}{t_f}\]

**FIG. I** NEUTRAL POINT AND FRICTION DIRECTIONS.
FIG. 2 STRIP, ROLL, AND ASSUMED DEFORMATION PATTERN.
FIG. 3 BOUNDARY OF VELOCITY DISCONTINUITY.

\[ R = \frac{1}{2} \frac{t_0}{\sin \alpha_2} \]

\[ v_0 \cos \theta \approx v_0 \]

\[ v_0 \sin \theta \]
\[
\frac{t_{K-1} + t_K}{4} - \frac{t_f}{2} = R_0 (1 - \cos \alpha_k)
\]

**FIG. 4** TRIANGULAR FIELD WITH LINEAR RIGID BODY MOTION.
FIG. 5 TRIANGULAR VELOCITY FIELD WITH ROTATIONAL RIGID BODY MOTION.
FIG. 6 TOTAL WEIGHTED POWER AS A FUNCTION OF THE NUMBER OF ELEMENTS.

\[
\frac{f_0}{R_0} = 0.01 \quad \frac{f_f}{f_0} = 0.7 \quad m = 0.2
\]

\[
\frac{y_f}{U} \text{ (LINEAR)} = 1.0478 \quad \frac{y_f}{U} \text{ (ROTATIONAL)} = 1.057
\]
FIG. 7 POWER CONSUMPTION AS A FUNCTION OF RELATIVE EXIT VELOCITY.
Fig. 7: Power consumption as a function of relative exit velocity.

Weighted power

Relative exit velocity

---

**Continuous Line**

- Linear: \( R_0 = 0.01 \)
- Rotational: \( \theta = 0.7 \)
- Ideal Power: \( m = 0.2 \)
FIG. 8 OPTIMAL RELATIVE EXIT VELOCITY AS A FUNCTION OF FRICTION.
FIG. 9 WEIGHTED POWER AS A FUNCTION OF FRICTION.
FIG. 9 WEIGHTED POWER AS A FUNCTION OF FRICTION.
FIG. 10 WEIGHTED POWER VS. RELATIVE THICKNESS
WEIGHTED POWER VS. RELATIVE THICKNESS

Fig. 10: Weighted Power vs. Relative Thickness

- Ideal Power
- Rotational Linear
- Continuous Linear
- Helical Linear
- Helical Rotational
- Helical Rotational (No) - N (Rotational)
- Helical Rotational (No) - N (Linear)

Parameters:
- $m = 0.2$
- $\frac{1}{h} = 0.7$

Velocity Field
FIG. 11 SPLIT END OF STEEL STRIP
FIG. 12 SPLIT END (ALLIGATORING) VS. SOUND FLOW.
REFERENCES


APPENDIX A

Derivation of Shear Power Losses Along \( \Gamma_{2k} \), \( \Gamma_{1k} \) for the Triangular Velocity Field with Rotational Rigid Body Motion

I. Equation for the surface \( \Gamma_{2k} \), \( \Gamma_{1k} \)

Work done by Avitzur and Pachla\(^1\) reveal that the surface of velocity discontinuity between a linear and rotating rigid body is a circle. For the specific case where the rotating rigid body revolves about the center of the coordinate system and the direction of the linear motion is perpendicular to the \( \phi = 0 \) axis, Eq. (4.2.1.2) from Ref. [16] can be simplified to the form of

\[
(R \cos \vartheta + \frac{v_1}{\omega_k})^2 + R^2 \sin^2 \vartheta = R_1^2
\]  \hspace{1cm} (2-14)

where \( R_1 \) is the radius of this circle with its center located at \((- \frac{v_1}{\omega_k}, 0)\) in \((R, \vartheta)\) coordinates. \( v_1 \) is the magnitude of the velocity of the linear rigid body and \( \omega_k \) is the angular velocity of the rotating rigid body.

For the \( \Gamma_{2k} \) surface, the velocity \( v_1 \) is defined as \( v_{k-1} \), in the linear direction, of magnitude expressed by Eq. (2-9). One endpoint of the surface intersects the roll at coordinates \((R_0, \kappa_{k-1})\). Substituting Eq. (2-9) and the point \((R_0, \kappa_{k-1})\) into Eq. (2-14) yields

\[
R_{2k} = \sqrt{R_0^2 + 2R_0 \frac{v_f}{t_f} \cos \kappa_{k-1} + \frac{v_f}{t_{f_{k-1}}} \kappa_{k-1}} \hspace{1cm} (A-1)
\]
Replacing Eq. (A-1) back into the general expression of Eq. (2-14) produces

\[
(R \cos \theta + \frac{v_f t_f}{t_{k-1} \omega_k})^2 R^2 \sin^2 \theta = R_o^2 + 2 R_o \frac{v_f t_f}{t_{k-1} \omega_k} \cos \alpha_{k-1} + \left(\frac{v_f t_f}{t_{k-1} \omega_k}\right)^2
\]  
(A-2)

In order to fix \( \omega_k \) and thus the center of this circle, a second point is needed. Along the plane of symmetry of the strip, this point has coordinates \((R_k, \theta_k)\) where \(R_k\) is given by Eq. (2-17) and

\[
\cos \theta_k = \frac{t_f/2 + R_o}{R_k}
\]  
(A-3)

Placing the coordinates \((R_k, \theta_k)\) back into Eq. (A-2) and simplifying

\[
R_k^2 - R_o^2 = 2 \frac{v_f t_f}{t_{k-1} \omega_k} (R_o \cos \alpha_{k-1} - R_k \cos \theta_k)
\]  
(A-4)

solving for \( \omega_k \)

\[
\omega_k = 2 \frac{v_f t_f}{t_{k-1}} \left[ \frac{R_o \cos \alpha_{k-1} - R_k \cos \theta_k}{R_k^2 - R_o^2} \right]
\]  
(A-5)
Substituting Eq. (A-3) for \( \cos \alpha_k \) and Eq. (2-19) for \( \cos \alpha_{k-1} \) into Eq. (A-5)

\[
\omega_k = \frac{v_f t_f}{R_0^2 - R_k^2}
\]

(A-6)

This leads to

\[
R_{12k} = \sqrt{R_0^2 + 2 \frac{R_0}{t_{k-1}} (R_0^2 - R_k^2) \cos \alpha_{k-1} + \left( \frac{R_0^2 - R_k^2}{t_{k-1}} \right)^2}
\]

(A-7)

Applying this same approach to the \( \gamma_{1k} \) surface

\[
R_{11k} = \sqrt{R_0^2 + 2 \frac{R_0}{t_k} (R_0^2 - R_k^2) \cos \alpha_k + \left( \frac{R_0^2 - R_k^2}{t_k} \right)^2}
\]

(A-8)

II. Derivation of Shear Losses Along \( \gamma_{2k} \), \( \gamma_{1k} \)

Further work by Avitzur and Pachla\(^\text{16}\) reveal that for the assumed velocity field, the shear power losses along the \( \gamma_{2k} \) surface follows Eq. (4.4.3) from Ref. [16],

\[
W_{s,k} = \frac{\sigma_0}{\sqrt{3}} \frac{R_i^2}{v_1} \left[ \omega_k \left\{ \sin^{-1} \left[ -\frac{X_A - \omega_k}{R_i} \right] - \sin^{-1} \left[ \frac{X_B + \omega_k}{R_i} \right] \right\} \right]
\]

(3-23)

where \( X_A \) and \( X_B \) are the two points through which the shear losses are to be calculated. Translating from cylindrical coordinates to cartesian
coordinates, \( x_A \) and \( x_B \) for the \( r_{2k} \) surface are

\[
x_A = R_0 \cos \alpha_{k-1} \quad (A-9)
\]

along the surface of the roll and

\[
x_B = R_k \cos \theta_k \quad A-10)
\]

on the plane of symmetry of the strip. Replacing \( \cos \alpha_{k-1} \) in Eq. (A-9) by Eq. (2-19) and \( \cos \theta_k \) in Eq. (A-10) by Eq. (A-3), the two endpoints simplify to

\[
x_A = R_0 - \frac{t_{k-1}}{2} + \frac{t_f}{2}
\]

\[
x_B = R_0 + \frac{t_f}{2} \quad (A-11)
\]

Again, for the \( r_{2k} \) surface, the velocity \( v_1 \) is defined by \( v_{k-1} \), expressed by Eq. (2-9). Employing Eq. (A-6) for \( \omega_k \), the ratio \( v_{k-1}/\omega_k \) becomes

\[
\frac{v_1}{\omega_k} = \frac{p^2 - R_k^2}{t_{k-1}} \quad (A-12)
\]

which is defined in the nomenclature of the present work on strip rolling as \( \omega_{k-1} \).
Placing Eq. (A-11) for $x_A$, $x_B$ and $W_{k-1}$ for the ratio $v_{k-1}/w_k$, the shear losses along the $r_{2k}$ surface reduce to

$$W_{s_{r_{2k}}} = \frac{\sigma_0 v_f t_f}{3} \left\{ \frac{R_{r_{2k}}^2}{R_0 - R_k^2} \sin^{-1} \left[ \frac{-R_0 + \frac{t_{k-1}}{2} - \frac{t_f}{2} - W_{k-1}}{R_{r_{2k}}^2} \right] \right\}$$

(A-13)

Applying this same approach to the $r_{1k}$ surface

$$W_{s_{r_{1k}}} = \frac{\sigma_0 v_f t_f}{3} \left\{ \frac{R_{r_{1k}}^2}{R_0 - R_k^2} \sin^{-1} \left[ \frac{-R_0 + \frac{t_k}{2} - \frac{t_f}{2} - W_k}{R_{r_{1k}}^2} \right] \right\}$$

(A-14)

These two equations are equivalent to Eqs. (3.24) and (3.25) respectively.
APPENDIX B

Derivation of Shear Power Losses Along $\tau_{2k}, \tau_{1k}$ for the Triangular Velocity Field with Linear Rigid Body Motion

Work done by Avitzur and Pachla shows an expression for the shear power losses along the surface of velocity discontinuity between two linear velocity fields. For the specific case where the direction of one linear motion is parallel to the x-axis, the power losses as defined by Eq. (3.4.3) of Ref. [16] simplify to

$$W_{sl} = \frac{\sigma_0}{\sqrt{3}} v_1 \left[ \frac{1 + \left( \frac{v_2}{v_1} \right)^2 - 2 \frac{v_2}{v_1} \cos \alpha}{1 - \frac{v_2}{v_1} \cos \alpha} \right] (X_B - X_A) \quad (3-10)$$

where $v_1$ is the magnitude of the velocity which is parallel to the x-axis. Likewise, $v_2$ is the magnitude of the velocity in the other zone. $\alpha$ is the angle between the two velocities and $X_A, X_B$ are the x-coordinate endpoints of the surface of velocity discontinuity.

Approaching the $\tau_{2k}$ surface, the velocity is linear, flowing in a direction parallel to the x-axis. So, $v_1$ in Eq. (3-10) is equivalent to $v_{k-1}$ with a magnitude given by Eq. (2-9). The magnitude of the velocity within the kth element, $v$, is equivalent to $v_2$ and is derived through the principles of volume constancy, resulting in Eq. (2-11).

From observation, the angle between the two velocities is $\alpha_k$. 

57
Furthermore, the $x$-coordinate endpoints of the $\Gamma_{2k}$ surface are

$$X_A = R_o \sin \alpha_k + x_k$$  \hspace{2cm} (B-1)$$

along the plane of symmetry of the strip and

$$X_B = R_o \sin \alpha_k + L_k$$  \hspace{2cm} (B-2)$$

along the surface of the roll.

$R_o$ is the radius of the roll. $\alpha_k$ is the angle between the intersection of the roll with the $\Gamma_{1k}$ surface and the axis of symmetry of the rolls. $x_k$ is the apex position of the element $k$ along the plane of symmetry of the strip. $L_k$ is the projected length of element $k$.

Replacing Eqs. (2-9), (2-11), (B-1), and (B-2) in Eq. (3-10)
Simplifying

\[
W_{sr}^{2k} = \frac{\sigma_0 v_f t_f}{\sqrt{3} t_{k-1}} \left\{ \frac{\sin^2(\beta_{2k} - \alpha_k) + \sin^2\beta_{2k} - 2 \sin \beta_{2k} \cos \alpha_k \sin(\beta_{2k} - \alpha_k)}{\sin(\beta_{2k} - \alpha_k)} \right\} \frac{(L_k - x_k)}{\sin^2\beta_{2k} \cos^2\alpha_k} \]

(B-4)

Noting that

\[
\sin(\beta_{2k} - \alpha_k) = \sin \beta_{2k} \cos \alpha_k - \sin \alpha_k \cos \beta_{2k} \quad (B-5)
\]

Eq. (B-4) after expanding the terms and simplifying, becomes

\[
W_{sr}^{2k} = \frac{\sigma_0 v_f t_f}{\sqrt{3} t_{k-1}} \left\{ \frac{\sin^2\beta_{2k} - \sin^2\beta_{2k} \cos^2\alpha_k + \sin^2\alpha_k \cos^2\beta_{2k}}{\sin^2\beta_{2k} \cos^2\alpha_k - \sin^2\alpha_k \cos^2\beta_{2k}} \right\} \frac{(L_k - x_k)}{\sin^2\alpha_k \cos^2\beta_{2k}} \]

(B-6)

Factoring out \(\sin^2\beta_{2k}\) and noting that

\[
\sin^2\alpha_k = 1 - \cos^2\alpha_k \quad (B-7)
\]
Eq. (B-6) reduces to

\[
W_{s_{2k}} = \frac{v_f}{3} \frac{t_f}{t_{k-1}} \left\{ \frac{\sin^2 2k \sin^2 \alpha_k + \cos^2 2k \sin^2 \alpha_k}{\sin 2k \cos \alpha_k - \sin \alpha_k \cos 2k} - \sin \alpha_k \cos 2k \right\} (L_k - x_k)
\]  

(B-8)

Factoring out \( \sin^2 \alpha_k \) and using the identity

\[ \sin^2 2k + \cos^2 2k = 1 \]  

(B-9)

reduces Eq. (B-8) to

\[
W_{s_{2k}} = \frac{v_0}{3} \frac{v_f}{t_f} \frac{t_f}{t_{k-1}} \left\{ \frac{L_k - x_k}{\cos^2 2k \sin^2 2k (\cot^2 2k - \cot^2 \alpha_k)} \right\}
\]  

(B-10)

From the geometry of the problem

\[
\cos \beta_{2k} = \frac{L_k - x_k}{(L_k - x_k)^2 + (t_{k-1}/2)^2}
\]  

(B-11)

\[
\sin \beta_{2k} = \frac{t_{k-1}/2}{(L_k - x_k)^2 + (t_{k-1}/2)^2}
\]

Defining

\[ L_k = \left( \frac{t_{k-1}}{2} - \frac{t_k}{2} \right) \cot \alpha_k \]  

(B-12)
simplifies the shear power losses along $\Gamma_{2k}$ surface to

$$
W_{S_2} = \frac{\sigma_0 v_f t_f}{\sqrt{3}} \left[ \frac{(L_k - x_k)^2 + \left(\frac{t_k-1}{2}\right)^2}{x_k + \frac{t_k}{2} \cot \alpha_k} \right]
$$  \hspace{1cm} (3-11)

Applying this same approach to the $\Gamma_{1k}$ surface

$$
v_1 = v_k \hspace{1cm} \text{(Eq. 2-12)}
$$

$$
v_2 = v \hspace{1cm} \text{(Eq. 2-13)}
$$

$$
X_A = R_0 \sin \alpha_k
$$

$$
X_B = R_0 \sin \alpha_k + x_k
$$

Placing these equations into Eq. (3-10), noting the appropriate identities and geometry, the shear power losses along $\Gamma_{1k}$ are expressed as

$$
W_{S_1} = \frac{\sigma_0 v_f t_f}{\sqrt{3}} \left[ \frac{x_k^2 + \left(\frac{t_k}{2}\right)^2}{\frac{t_k}{2} \cot \alpha_k + x_k} \right]
$$  \hspace{1cm} (3-12)
Determination of \( (x_k/R_o) \) for the Triangular Velocity Field with Linear Rigid Body Motion

The total shear losses along the \( \Gamma_{2k} \) and \( \Gamma_{1k} \) surfaces are

\[
\dot{\Omega}_k = \dot{\Omega}_{1k} + \dot{\Omega}_{2k}
\]

\[
\frac{2\sigma_0 v_T t_f}{\sqrt{3}} \frac{1}{\frac{t_k}{2} \cot \alpha_k + x_k} \left[ \frac{x_k + \left( \frac{t_k}{2} \right)^2}{t_k} \right] + \frac{1}{\frac{t_k}{2} \cot \alpha_k + x_k} \left[ \frac{(L_k - x_k)^2}{t_k} \right]
\]

For further use let the following differentiation be performed

\[
\frac{\partial \Omega_k}{\partial x_k} = \frac{2\sigma_0 v_T t_f}{\sqrt{3}} \left\{ \frac{1}{\frac{t_k}{2} \cot \alpha_k + x_k} \left[ \frac{x_k + \left( \frac{t_k}{2} \right)^2}{t_k} \right] + \frac{1}{\frac{t_k}{2} \cot \alpha_k + x_k} \left[ \frac{(L_k - x_k)^2}{t_k} \right] \right\}
\]

\[
= \frac{2\sigma_0 v_T t_f}{\sqrt{3}} \frac{1}{\frac{t_k}{2} \cot \alpha_k + x_k} \left\{ \frac{x_k - L_k - x_k}{t_k} \left( \frac{t_k}{2} \cot \alpha_k + x_k \right) \right\}
\]
Likewise, the expression for the friction losses along the element \( k \) can be simplified to

\[
W_{s_3k} = \frac{2}{\sqrt{3}} \sigma_0 m U \left( 1 - \frac{v_f t_f}{U t_k} \right) \frac{1}{\cos \alpha_k - \frac{x_k}{t_k} \sin \alpha_k} \frac{t_{k-1} - t_k}{2 \sin \alpha_k}
\]  
(C-3)

For further use, let the following differentiation be performed.

(I). If \( (1 - \frac{v_f t_f}{U t_k} \frac{1}{\cos \alpha_k + \frac{x_k}{t_k} \sin \alpha_k}) > 0 \)

Then,

\[
\frac{\partial W_{s_3k}}{\partial x_k} = \sigma_0 m U \left( 1 - \frac{v_f t_f}{U t_k} \right) \frac{1}{\cos \alpha_k + \frac{x_k}{t_k} \sin \alpha_k} \frac{t_{k-1} - t_k}{2 \sin \alpha_k} \frac{t_k}{t_f} \left( \cos \alpha_k + \frac{x_k}{t_k} \sin \alpha_k \right)
\]  
(\text{C-4})

\[
= \frac{2 \sigma_0 m U t_f}{\sqrt{3}} \left( \frac{t_{k-1}}{t_k} - 1 \right) \frac{v_f}{U} \frac{t_k/4}{\sin^2 \alpha_k \left[ \frac{t_k}{2} \cot \alpha_k + x_k \right]^2}
\]
(II) If 
\[(1 - \frac{v_f t_f}{U t_k}) \frac{1}{\cos \alpha_k + 2 \frac{x_k}{t_k} \sin \alpha_k} < 0\]

Then, 
\[\frac{3W_s}{3k} = \frac{\sigma_0 m U t_f}{\sqrt{3}} \left( \frac{v_f t_f}{U t_k} \frac{1}{\cos \alpha_k + 2 \frac{x_k}{t_k} \sin \alpha_k} - 1 \right) \left( \frac{t_k-1}{t_f} - \frac{t_k}{t_f} \right) \text{ (C-5)}\]

and 
\[\frac{3W_s}{3k}\]
is given by the right hand expression of Eq. (58) with

\[\frac{3W_s}{3x_k}\]
a negative sign).

The position of \(x_k\) which minimizes the total power consumption at

element (k) can be found by differentiation of the total power (of

the kth element) and setting the result equal to zero as follows

\[\frac{3W}{3x_k} = \frac{3W_s}{3x_k} + \frac{3W_s}{3x_k} \text{ (C-6)}\]

Substitution of Eq. (C-2) and (C-4) into Eq. (C-6) leads to two
derivations:

(I) If 
\[(1 - \frac{v_f t_f}{U t_k}) \frac{1}{\cos \alpha_k + 2 \frac{x_k}{t_k} \sin \alpha_k} > 0\]
\[
\frac{\delta y_k}{\delta x_k} = \frac{2}{\sqrt{3}} \frac{a \cdot \delta f}{\frac{t_k}{2} \cot \alpha_k} \left\{ 2 \left( \frac{x_k}{t_k} - \frac{L_k}{r_{k-1}} \right) \left( \frac{t_k}{2} \cot \alpha_k \cdot x_k \right) - \frac{t_k}{t_{k-1}} - \frac{(t_{k-1})^2}{\left( \frac{t_k}{2} \right)^2} \right\}
\]

\[
+ \frac{2}{\sqrt{3}} \frac{a \cdot \delta f}{\left( \frac{t_k}{2} \cot \alpha_k + x_k \right)^2 \sin \alpha_k} \cdot \frac{t_k}{t_{k-1}} - \frac{m}{4} = 0
\]

\[
\frac{2}{\sqrt{3}} \frac{a \cdot \delta f}{\left( \frac{t_k}{2} \cot \alpha_k + x_k \right)^2} \left\{ \left( \frac{t_k}{R_o} \right)^2 + \left( \frac{t_k}{t_{k-1}} \right)^2 \cot \alpha_k - \left( \frac{t_k}{R_o} \right)^2 \cot \alpha_k \right\}
\]

\[
\frac{2}{\sqrt{3}} \frac{a \cdot \delta f}{\left( \frac{t_k}{2} \cot \alpha_k + x_k \right)^2} \left\{ \left( \frac{t_k}{R_o} \right)^2 + \left( \frac{t_k}{t_{k-1}} \right)^2 \cot \alpha_k - \left( \frac{t_k}{R_o} \right)^2 \cot \alpha_k \right\}
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]

\[
\frac{R_o}{t_k} \left( 1 + \frac{t_k}{t_{k-1}} \right) \left( \frac{x_k}{R_o} \right)^2 + \left( 1 + \frac{t_k}{t_{k-1}} \right) \cot \alpha_k \left( \frac{x_k}{R_o} \right) - \frac{t_k}{4 R_o} \left( 1 + \frac{t_k}{t_{k-1}} \right) - \frac{t_k}{4 R_o} \left( 1 - \frac{t_k^2}{t_{k-1}} \right) \cot \alpha_k
\]
\[
\begin{align*}
\frac{x_k}{R_o \text{ optimal}} &= -\cot \frac{\alpha}{\tan \frac{\alpha}{\tan \frac{t_{k-1}}{t_k}} - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \right)^2 \right) \right)
\end{align*}
\]

\[
\begin{align*}
&= -\frac{t_k}{2R_o} \left[-\cot \frac{\alpha}{\tan \frac{\alpha}{\tan \frac{t_{k-1}}{t_k}} - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 - \frac{t_{k-1}}{t_k} \cdot \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \right)^2 \right) \right) \\
&= -\frac{t_k}{2R_o} \left[-\cot \frac{\alpha}{\tan \frac{\alpha}{\tan \frac{t_{k-1}}{t_k}} + \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 - \frac{1}{1+\lambda} \right) \right]
\end{align*}
\]

where \( \lambda = \frac{t_k}{t_{k-1}} \)

\[
\begin{align*}
(II) \quad \text{if} \quad (1 - \frac{v}{U} \cdot \frac{t_i}{t_k} = \frac{1}{\cos \alpha + \frac{x_k}{t_k} \cdot \sin \alpha}) < 0
\end{align*}
\]

\[
\begin{align*}
\frac{x_k}{R_o \text{ optimal}} &= \frac{t_k}{2R_o} \left[-\cot \frac{\alpha}{\tan \frac{\alpha}{\tan \frac{t_{k-1}}{t_k}} + \frac{1}{\sin \frac{\alpha}{\sin \frac{t_{k-1}}{t_k}}} \left(1 + \frac{1}{1+\lambda} \right) \right]
\end{align*}
\]
The total power consumed around the element \( k \) is found by the summation of Eqs. (C-1) and (C-3) to read

\[
J^* = \frac{J^*_k}{\omega f} = \frac{v_f}{u} \left[ \frac{1}{2} \frac{t_{k-1}}{R_o} \cota_k \cdot x_k \cdot \frac{t_k}{R_o} - \frac{1}{R_o} \right]
\]

\[
\left\{ \frac{x_k}{R_o}^2 + \frac{1}{4} \left( \frac{t_k}{R_o} \right)^2 \frac{L_k}{R_o} \cdot x_k^2 + \frac{1}{4} \left( \frac{t_k-1}{R_o} \right)^2 \left( \frac{t_k}{R_o} \right) \right\}
\]

\[
+ \frac{m}{\sin \alpha_k} \left| 1 - \frac{v_f}{u} \frac{t_f/R_o}{t_k/R_o} \cos \alpha_k + \frac{1}{2} \frac{x_k/R_o}{\sin \alpha_k} \right|
\]

Substitution of values of \( x_k/R_o \) from Eqs. (C-10) and (C-12) lead to Eq. (3-21).
VITA

William Arthur Gordon, son of Mr. & Mrs. Walter F. Irmscher, was born on July 4, 1960 in Bronxville, New York. He graduated from Tuckahoe High School in Eastchester, New York in 1978 and in the Fall of that year matriculated in the Engineering Program at Lafayette College. He received a B.S. in Metallurgical Engineering from Lafayette in 1982. While in his junior year, he was elected to the Tau Beta Pi Engineering Honor Society and was awarded the Charles Duncan Fraser Prize in Metallurgy. Upon graduation, he was granted Honors in Metallurgical Engineering for his thesis work entitled, "Analysis of Central Bursts in Forgings."

After leaving Lafayette, he spent the summer of 1982 employed by Bethlehem Steel Corporation as an Assistant Research Engineer in the Deformation Processes Division. Since September, 1982 he has worked as a Research Assistant while pursuing his M.S. in Metallurgy and Materials Engineering at Lehigh University. He is presently a member of the Institute for Metal Forming.